Math 444/539 : Geometric Topology
Problem Set 2

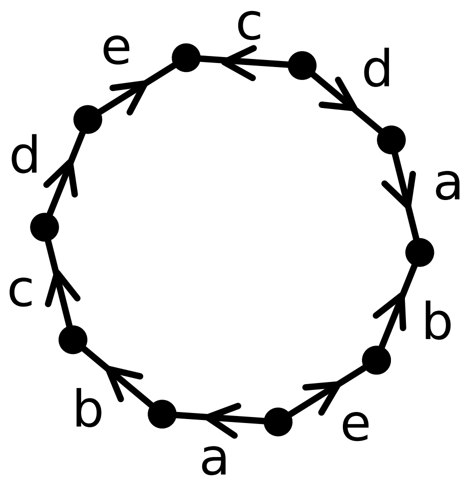
Everyone has to do problems 1-6. People enrolled in MATH 444 can do either 7 or 8, while people enrolled in MATH 539 (ie mathematics grad students) must do both 7 and 8.

1. Fix $a \in \mathbb{R}$, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. Set $X = \{p \in \mathbb{R}^n \mid f(p) = a\}$. Assume that for all $p \in X$, there exists some $1 \leq i \leq n$ such that $\frac{\partial f}{\partial x_i}$ is nonzero at p. Prove that X is a manifold. Hint : implicit function theorem. Of course, this generalizes the fact that the $(n-1)$-sphere in $\mathbb{R}^n$ is a manifold...

2. Call $\mathrm{Int}(D^2) \times S^1$ an *open solid torus*. Prove that there exists a torus T embedded in S^3 such that $S^3 \setminus T$ is the disjoint union of 2 open solid tori (hint : think of S^3 as the set of all points $(x, y, z, w) \in \mathbb{R}^4$ such that $x^2 + y^2 + z^2 + w^2 = 1$. How can you find a copy of $S^1 \times S^1$ in this?).

3. Prove that $\mathbb{R}P^n$ is an n-manifold.

4. What surface is obtained by identifying the sides of a 10-gon as shown in the following picture?



5. It was known to the ancient Greeks that there are only five regular polyhedra, namely the regular tetrahedron, cube, octahedron, dodecahedron, and icosahedron. Prove this by considering subdivisions of the sphere into n-gons (n fixed) such that exactly m edges meet at each vertex (M fixed, $m, n \geq 3$). Hint : Euler characteristic.

6. Prove that it is not possible to subdivide the surface of a sphere into regions, each of which has 6 sides and such that distinct regions have no more than one side in common.

7. Let X be a surface obtained by gluing Euclidean polygons together. More precisely, assume that there is a finite set $\{t_1, \ldots, t_n\}$ of disjoint polygons in $\mathbb{R}^2$. Let $(s_1, s_1'), \ldots, (s_k, s_k')$ be ordered pairs with the following properties.

   - Each s_i and s_i' is a side of one of the t_j.

- The list $s_1, s_1', \ldots, s_k, s_k'$ contains each side of each polyhedron in $\{t_1, \ldots, t_n\}$ exactly once.
- For all $1 \leq i \leq k$, the sides s_i and s_i' have the same length and a linear homeomorphism $\phi_i : s_i \to s_i'$ has been chosen.

Then X is the quotient of $t_1 \sqcup \cdots \sqcup t_n$ by the relation that identifies $x \in s_i$ with $\phi_i(x) \in s_i'$ for all $1 \leq i \leq k$.

(a) Prove that X is a compact topological surface.

(b) For each vertex v of X, let $\kappa(v)$ equal 2π minus the sum of the angles of the corners of the polygons abutting v (this is often known as the "curvature" at v). Prove the following formula, which is a discrete analogue of the famous Gauss–Bonnet theorem.

$$\sum_{v \in X^{(0)}} \kappa(v) = 2\pi\chi(X).$$

8. The Jordan curve theorem is as follows.

Theorem. *Let $f : [0,1] \to \mathbb{R}^2$ be a continuous map such that $f(0) = f(1)$ and such that $f|_{(0,1)}$ is injective. Then $\mathbb{R}^2 \setminus \mathrm{Im}(f)$ has exactly 2 connected components.*

This is not an easy theorem to prove; however, in this problem you will prove a special case of it. Say that a function $f : [0,1] \to \mathbb{R}^2$ is *piecewise linear* if there exist $0 = a_1 < a_2 < \cdots < a_k = 1$ such that $f|_{[a_i, a_{i+1}]}$ is linear (ie there exist constants $c, c', d, d' \in \mathbb{R}$ such that for $x \in [a_i, a_{i+1}]$ we have $f(x) = (cx + d, c'x + d')$).

Now let $f : [0,1] \to \mathbb{R}^2$ be a continuous piecewise linear map such that $f(0) = f(1)$ and such that $f|_{(0,1)}$ is injective. Let $0 = a_1 < a_2 < \cdots < a_k = 1$ be the subdivision coming from the piecewise linearity of f.

(a) Prove that $\mathbb{R}^2 \setminus \mathrm{Im}(f)$ has at most 2 path components.

(b) Prove that $\mathbb{R}^2 \setminus \mathrm{Im}(f)$ has at least 2 path components. Here's an outline. It is enough to find disjoint open sets U and V such that $\mathbb{R}^2 \setminus \mathrm{Im}(f) = U \cup V$. Consider $p \in \mathbb{R}^2 \setminus \mathrm{Im}(f)$. Say that a ray emanating from p is *generic* if it intersects $\mathrm{Im}(f)$ in finitely many places and does not intersect $f(a_i)$ for any $1 \leq i \leq k$. Prove first that every $p \in \mathbb{R}^2 \setminus \mathrm{Im}(f)$ has generic rays emanating from it. Let r be a generic ray emanating from p. Say that p is even if r intersects $\mathrm{Im}(f)$ in an even number of points and odd otherwise. Prove next that this is independent of r. Now define $U = \{q \in \mathbb{R}^2 \setminus \mathrm{Im}(f) \mid q \text{ is even}\}$ and $V = \{q \in \mathbb{R}^2 \setminus \mathrm{Im}(f) \mid q \text{ is odd}\}$. Prove that U and V are disjoint open sets whose union is $\mathbb{R}^2 \setminus \mathrm{Im}(f)$.