Math 444/539 : Geometric Topology
Problem Set 4

Students enrolled in Math 539 must do all the problems. Students enrolled in Math 444 can omit problem 6.

Problems :

1. Let X be a path-connected topological space with **abelian** fundamental group. Fix two points $p, q \in X$. Recall that $\varphi_\gamma : \pi_1(X, q) \to \pi_1(X, p)$ is the homomorphism associated to an equivalence class γ of paths from p to q. Prove that if γ and γ' are two paths from p to q, then $\varphi_\gamma = \varphi_{\gamma'}$.

2. Let X be a topological space, let $p, q \in X$ be two points, and let f and g be two paths from p to q. Prove that f is equivalent to g if and only if $f \cdot \overline{g}$ is equivalent to the constant path e_p.

3. Let M be a Möbius strip. Find an embedded circle C in M such that M deformation retracts to C.

4. Let S be an orientable genus g surface. Find some $A \subset S$ with the following properties.

   - A is homeomorphic to a graph.
   - There exists a point $p \in S$ such that $S \setminus \{p\}$ deformation retracts onto A.

   Hint : Think of S as a $4g$-gon with sides identified.

5. Let X be a topological space. Prove that the following three conditions are equivalent.

   (a) Every map $S^1 \to X$ is homotopic to a constant map.

   (b) For every map $f : S^1 \to X$, there exists a map $g : D^2 \to X$ such that $g|_{\partial D^2} = f$.

   (c) For all $p \in X$, we have $\pi_1(X, p) = 1$.

   Deduce that a space is 1-connected if and only if all maps $S^1 \to X$ are homotopic. I want the emphasize that in this problem, "homotopic" means "homotopic without regards to basepoints".

6. Let G be a topological group. Let $e \in G$ be the identity element. Prove that $\pi_1(G, e)$ is abelian. Hint : in addition to the multiplication of loops $\cdot$ in $\pi_1(G, e)$, the group structure of G gives another way of multiplying loops. Namely, for loops f and g based at e, we can define $f * g$ to be the loop $t \mapsto f(t)g(t)$. The first step is to prove that the loop $f * g$ is equivalent to the loop $g \cdot f$.

7. Let X be a topological space and let $\{U_\alpha\}$ be an open covering of X with the following properties.

   (a) There exists a point $p \in X$ such that $p \in U_\alpha$ for all α.

   (b) Each U_α is simply-connected.

   (c) For $\alpha \neq \beta$, the set $U_\alpha \cap U_\beta$ is path-connected.

Prove that X is simply-connected. Hint : consider $\gamma \in \pi_1(X, p)$. Prove that we can write $\gamma = \gamma_1 \cdots \gamma_k$, where $\gamma_i \in \pi_1(X, p)$ can be realized by a loop based at p that lies entirely inside one of the U_α. The notion of the *Lebesgue number* of a covering from point-set topology will be useful here.

8. Using the previous problem, prove that S^n is simply-connected for $n \geq 2$.