Math 444/539 : Geometric Topology
Problem Set 4

Students enrolled in Math 539 must do all the problems. Students enrolled in Math 444 can omit problem 6.

Problems :

1. Let \( X \) be a path-connected topological space with abelian fundamental group. Fix two points \( p, q \in X \). Recall that \( \varphi_\gamma : \pi_1(X, q) \to \pi_1(X, p) \) is the homomorphism associated to an equivalence class \( \gamma \) of paths from \( p \) to \( q \). Prove that if \( \gamma \) and \( \gamma' \) are two paths from \( p \) to \( q \), then \( \varphi_\gamma = \varphi_{\gamma'} \).

2. Let \( X \) be a topological space, let \( p, q \in X \) be two points, and let \( f \) and \( g \) be two paths from \( p \) to \( q \). Prove that \( f \) is equivalent to \( g \) if and only if \( f \cdot g \) is equivalent to the constant path \( e_p \).

3. Let \( M \) be a M"obius strip. Find an embedded circle \( C \) in \( M \) such that \( M \) deformation retracts to \( C \).

4. Let \( S \) be an orientable genus \( g \) surface. Find some \( A \subset S \) with the following properties.
   - \( A \) is homeomorphic to a graph.
   - There exists a point \( p \in S \) such that \( S \setminus \{p\} \) deformation retracts onto \( A \).

   Hint : Think of \( S \) as a \( 4g \)-gon with sides identified.

5. Let \( X \) be a topological space. Prove that the following three conditions are equivalent.
   (a) Every map \( S^1 \to X \) is homotopic to a constant map.
   (b) For every map \( f : S^1 \to X \), there exists a map \( g : D^2 \to X \) such that \( g|_{\partial D^2} = f \).
   (c) For all \( p \in X \), we have \( \pi_1(X, p) = 1 \).

   Deduce that a space is 1-connected if and only if all maps \( S^1 \to X \) are homotopic. I want the emphasize that in this problem, “homotopic” means “homotopic without regards to basepoints”.

6. Let \( G \) be a topological group. Let \( e \in G \) be the identity element. Prove that \( \pi_1(G, e) \) is abelian. Hint : in addition to the multiplication of loops \( \cdot \) in \( \pi_1(G, e) \), the group structure of \( G \) gives another way of multiplying loops. Namely, for loops \( f \) and \( g \) based at \( e \), we can define \( f \ast g \) to be the loop \( t \mapsto f(t)g(t) \). The first step is to prove that the loop \( f \ast g \) is equivalent to the loop \( g \cdot f \).

7. Let \( X \) be a topological space and let \( \{U_\alpha\} \) be an open covering of \( X \) with the following properties.
   (a) There exists a point \( p \in X \) such that \( p \in U_\alpha \) for all \( \alpha \).
   (b) Each \( U_\alpha \) is simply-connected.
   (c) For \( \alpha \neq \beta \), the set \( U_\alpha \cap U_\beta \) is path-connected.
Prove that $X$ is simply-connected. Hint: consider $\gamma \in \pi_1(X,p)$. Prove that we can write $\gamma = \gamma_1 \cdots \gamma_k$, where $\gamma_i \in \pi_1(X,p)$ can be realized by a loop based at $p$ that lies entirely inside one of the $U_\alpha$. The notion of the Lebesgue number of a covering from point-set topology will be useful here.

8. Using the previous problem, prove that $S^n$ is simply-connected for $n \geq 2$. 