Math 444/539 : Geometric Topology
Problem Set 5

Everyone should do all the problems.

Problems:

1. Define \( f : S^1 \times I \to S^1 \times I \) by \( f(\theta, s) = (\theta + 2\pi s, s) \), so \( f \) restricts to the identity on the two boundary circles of \( S^1 \times I \). Show that \( f \) is homotopic to the identity by a homotopy that is stationary on one of the boundary circles, but not by any homotopy that is stationary on both boundary circles. Hint: consider what \( f \) does to the path \( s \mapsto (\theta_0, s) \) for fixed \( \theta_0 \).

2. (a) Prove that the Borsuk-Ulam theorem does not hold for \( T^2 \). More specifically, find a function \( f : S^1 \times S^1 \to \mathbb{R}^2 \) such that there does not exist a point \((x, y) \in S^1 \times S^1 \) such that \( f(x, y) = f(-x, -y) \).

(b) Recall that we used the Borsuk-Ulam theorem to show that \( S^2 \) cannot be embedded in \( \mathbb{R}^2 \). Prove that an orientable genus \( g \) surface cannot be embedded in \( \mathbb{R}^2 \) for \( g \geq 1 \). Hint: by part a, there is no version of the Borsuk-Ulam theorem available, so you’ll have to try something else. I would suggest assuming that the surface can be embedded, and then using the Jordan curve theorem in a clever way to derive a contradiction.

3. Let \( A_1, A_2, \) and \( A_3 \) be three polyhedra in \( \mathbb{R}^3 \). Use the Borsuk-Ulam theorem to show that there exists a plane \( P \subset \mathbb{R}^2 \) that divides each \( A_i \) into two pieces of equal volume.

4. Show that if \( \phi : \pi_1(S^1, 1) \to \pi_1(S^1, 1) \) is any homomorphism, then there exists some map \( f : S^1 \to S^1 \) such that \( f_* = \phi \). Remark: in particular, \( f(1) = 1 \).

5. Prove that there are infinitely many non-homotopic retractions \( S^1 \vee S^1 \to S^1 \). Remark: we do not yet know the fundamental group of \( S^1 \vee S^1 \) – you are not allowed to read ahead and use that calculation. You must prove every fact you need about the fundamental group of \( S^1 \vee S^1 \) using things covered in the class up to this point.

6. Consider a map \( f : S^1 \to S^1 \). Pick some path \( \gamma \) from \( f(1) \in S^1 \) to \( 1 \in S^1 \). We therefore get an induced sequence of maps

\[
\mathbb{Z} = \pi_1(S^1, 1) \xrightarrow{f_*} \pi_1(S^1, f(1)) \xrightarrow{\phi} \pi_1(S^1, 1) = \mathbb{Z}.
\]

which we will denote \( \psi : \mathbb{Z} \to \mathbb{Z} \).

(a) Prove that \( \psi \) is multiplication by some integer \( n \).

(b) Prove that \( n \) is independent of the choice of path \( \gamma \).

This integer \( n \) is known as the degree of \( f \).

7. (a) Prove that if a map \( f : S^1 \to S^1 \) is nonsurjective, then its degree is 0.

(b) Prove that any map \( f : S^1 \to S^1 \) of degree different from 1 has a fixed point.