Math 444/539 : Geometric Topology
Problem Set 5

Everyone should do all the problems.

Problems :

1. Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy that is stationary on one of the boundary circles, but not by any homotopy that is stationary on both boundary circles. Hint : consider what f does to the path $s \mapsto (\theta_0, s)$ for fixed θ_0.

2. (a) Prove that the Borsuk-Ulam theorem does not hold for T^2. More specifically, find a function $f : S^1 \times S^1 \to \mathbb{R}^2$ such that there does not exist a point $(x, y) \in S^1 \times S^1$ such that $f(x, y) = f(-x, -y)$.

   (b) Recall that we used the Borsuk-Ulam theorem to show that S^2 cannot be embedded in $\mathbb{R}^2$. Prove that an orientable genus g surface cannot be embedded in $\mathbb{R}^2$ for $g \geq 1$. Hint : by part a, there is no version of the Borsuk-Ulam theorem available, so you'll have to try something else. I would suggest assuming that the surface can be embedded, and then using the Jordan curve theorem in a clever way to derive a contradiction.

3. Let A_1, A_2, and A_3 be three polyhedra in $\mathbb{R}^3$. Use the Borsuk-Ulam theorem to show that there exists a plane $P \subset \mathbb{R}^2$ that divides each A_i into two pieces of equal volume.

4. Show that if $\phi : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ is any homomorphism, then there exists some map $f : S^1 \to S^1$ such that $f_* = \phi$. Remark : in particular, $f(1) = 1$.

5. Prove that there are infinitely many non-homotopic retractions $S^1 \vee S^1 \to S^1$. Remark: we do not yet know the fundamental group of $S^1 \vee S^1$ – you are not allowed to read ahead and use that calculation. You must prove every fact you need about the fundamental group of $S^1 \vee S^1$ using things covered in the class up to this point.

6. Consider a map $f : S^1 \to S^1$. Pick some path γ from $f(1) \in S^1$ to $1 \in S^1$. We therefore get an induced sequence of maps

   $$\mathbb{Z} = \pi_1(S^1, 1) \xrightarrow{f_*} \pi_1(S^1, f(1)) \xrightarrow{\phi_\gamma} \pi_1(S^1, 1) = \mathbb{Z}.$$

   which we will denote $\psi : \mathbb{Z} \to \mathbb{Z}$.

   (a) Prove that ψ is multiplication by some integer n.

   (b) Prove that n is independent of the choice of path γ.

   This integer n is known as the *degree* of f.

7. (a) Prove that if a map $f : S^1 \to S^1$ is nonsurjective, then its degree is 0.

   (b) Prove that any map $f : S^1 \to S^1$ of degree different from 1 has a fixed point.