Math 444/539 : Geometric Topology
Problem Set 6

Everyone should do all the problems.

Problems :

1. Let G and H be nontrivial groups and let $\Gamma = G * H$. Prove that Γ has a trivial center and that if $x \in \Gamma$ satisfies $x^n = 1$ for some $n \geq 1$, then x is conjugate to an element of G or H.

2. Let $X \subset \mathbb{R}^n$ be a finite set of points. Assume that $n \geq 3$. Prove that $\pi_1(\mathbb{R}^n \setminus X) = 1$.

3. Let $X \subset \mathbb{R}^3$ be a set of n distinct lines through the origin. Calculate $\pi_1(\mathbb{R}^3 \setminus X)$.

4. Let X equal $T^2 \sqcup T^2$ modulo the equivalence relation that identifies the circles $S^1 \times 1$ in the two tori homeomorphically. Calculate $\pi_1(X)$.

5. Let $X = \cup_{n=1}^{\infty} X_n$, where $X_n \subset \mathbb{R}^2$ is the circle of center $(1/n, 0)$ and radius $1/n$. Let $p = (0,0)$. Prove that $\pi_1(X,p)$ is uncountable. Hint : construct a retraction $r_n : X \to X_n$, and thus a surjection $(r_n)_* : \pi_1(X,p) \to \pi_1(X_n,p) = \mathbb{Z}$. Combine the r_n together to get a map $R : \pi_1(X,p) \to \prod_{n=1}^{\infty} \pi_1(X_n,p)$. Prove that R is surjective.

6. Recall that $\pi_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$.

   (a) Consider $(n,m) \in \mathbb{Z} \oplus \mathbb{Z}$. Assume that n and m are relatively prime. Prove that the curve on T^2 representing the homotopy class of (n,m) can be chosen so that it has no self-intersections.

   (b) Prove the converse of part a. In more detail, consider $[\gamma] \in \pi_1(T^2)$, where γ has no self-intersections. Let $(n,m) \in \mathbb{Z} \oplus \mathbb{Z}$ be the associated pair of numbers. Prove that n and m are relatively prime. Hint : First use an Euler characteristic argument to show that γ does not separate T^2 unless $(n,m) = (0,0)$. Assume, therefore that $(n,m) \neq (0,0)$. Use one of the previous homework problems to deduce that there is a homeomorphism $\phi : T^2 \to T^2$ taking the loop corresponding to $(1,0)$ to γ. At this point, you can assume that ϕ fixes the basepoint (this is not hard to prove, but I won't make you do it). Conclude that there is an isomorphism $\phi_* : \mathbb{Z}^2 \to \mathbb{Z}^2$ with $\phi_*(1,0) = (n,m)$. Why does this imply that n and m are relatively prime?