Math 444/539 : Geometric Topology
Problem Set 7

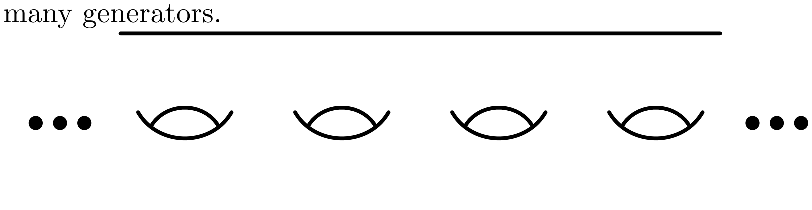
Everyone should do all the problems.

Problems :

1. Let $\Sigma_{g,n}$ be an oriented genus g surface with n boundary components. Assume that $g \geq 2$ and that $n \geq 1$. Prove that $\pi_1(\Sigma_{g,n})$ is a free group on $2g+n-1$ generators.

2. Let Σ_g be an oriented genus g surface. Assume that $g \geq 2$. Prove that $\pi_1(\Sigma_g)$ is not abelian. Hint : find a surjective homomorphism from $\pi_1(\Sigma_g)$ to the dihedral group of order 8.

3. Prove that the fundamental group of the following noncompact surface is free on infinitely many generators.

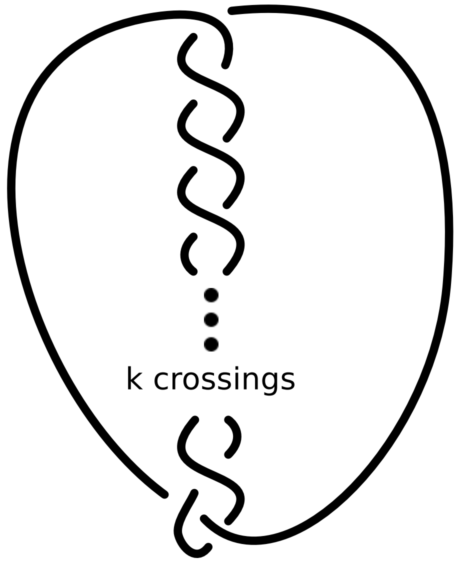


4. Consider the quotient space of a cube $I \times I \times I$ obtained by identifying each square face with the opposite square face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space X is a cell complex with two 0 cells, four 1 cells, three 2 cells, and one 3 cell. Using this structure, show that $\pi_1(X)$ is the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ of order 8.

5. Let $f : T^2 \to T^2$ be a map satisfying $f(p) = p$ for some point p. Since $\pi_1(T^2, p) \cong \mathbb{Z}^2$, we get an induced map $f_* : \mathbb{Z}^2 \to \mathbb{Z}^2$; ie a 2×2 integer matrix. Define M_f to be $T^2 \times I$ modulo the equivalence relation that identifies $(x, 1)$ with $(f(x), 0)$ (this is called the mapping torus of f). Compute $\pi_1(M_f)$ in terms of the above matrix.

6. Prove that the abelianization of the fundamental group of the complement of any tame knot is $\mathbb{Z}$.

7. (a) Computer a presentation for the fundamental group of the complement of the following knot.



(b) Show that this fundamental group also has a presentation with 2 generators and 1 relation.

8. (Challenge problem/extra credit). Let $f : I \to \mathbb{R}^3$ be a smooth simple closed curve. By this, I mean that f satisfies the following properties.

- f is differentiable and $f'(x) \neq 0$ for all x.
- $f(0) = f(1)$ and $f(x) \neq f(y)$ for $x, y \in (0, 1)$ distinct.
- $f'(0) = f'(1)$.

Consider the map $g : I \to S^1$ defined by $g(x) = \frac{f'(x)}{\|f'(x)\|}$. Prove that g is a generator for $\pi_1(S^1, g(0))$.