Math 444/539 : Geometric Topology
Problem Set 9

Everyone should do all the problems.

Problems :

1. For a covering space $\rho : \tilde{X} \to X$ and a subspace $A \subset X$, let $\tilde{A} = \rho^{-1}(A)$. Prove that $\rho|_{\tilde{A}} : \tilde{A} \to A$ is a covering space.

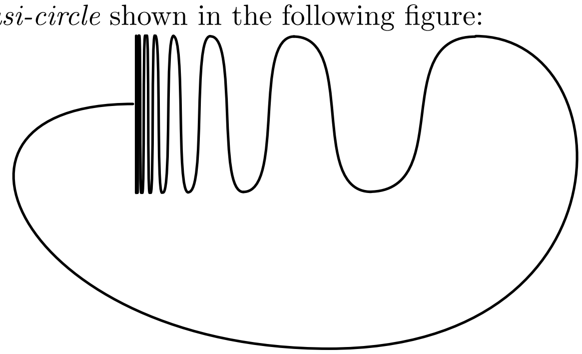
2. Show that if $\rho_1 : \tilde{X}_1 \to X_1$ and $\rho_2 : \tilde{X}_2 \to X_2$ are covering spaces, then so is $\rho_1 \times \rho_2 : \tilde{X}_1 \times \tilde{X}_2 \to X_1 \times X_2$.

3. Let $\rho : \tilde{X} \to X$ be a covering space with $\rho^{-1}(x)$ finite and nonempty for all $x \in X$. Prove that $\tilde{X}$ is compact Hausdorff if and only if X is compact Hausdorff.

4. Construct the universal cover of the following space:
$$X = \{x \in \mathbb{R}^3 \mid \|x\| = 1\} \cup \{(x,0,0) \in \mathbb{R}^3 \mid -1 \leq x \leq 1\}.$$

5. Let X be the subspace of $\mathbb{R}^2$ consisting of the four sides of the square $[0,1] \times [0,1]$ together with the segments of the vertical lines $x = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ inside the square. Show that for every covering space $\rho : \tilde{X} \to X$, there is some neighborhood of the left edge of X that lifts homeomorphically to $\tilde{X}$. Deduce that X has no simply-connected covering space.

6. Let Y be the *quasi-circle* shown in the following figure:



Thus Y is a closed subspace of $\mathbb{R}^2$ consisting of the union of the set
$$\{(x,y) \in \mathbb{R}^2 \mid 0 < x \leq 1,\ y = \sin(\frac{1}{x})\},$$
the set
$$\{(0,y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\},$$
and an arc joining the point $(0,0)$ to the point $(1, \sin(1))$. Collapsing the segment of Y in the y-axis to a point gives a quotient map $f : Y \to S^1$. Show that f does not lift to the universal covering space $\rho : \mathbb{R} \to S^1$ even though $\pi_1(Y) = 0$. Thus local path-connectedness of Y is a necessary hypothesis in the lifting criterion.

7. Let X be a path-connected, locally path-connected space with $\pi_1(X)$ a finite group. Prove that every map $f : X \to S^1$ is nullhomotopic.

8. Find all connected 2-sheeted and 3-sheeted coverings of the wedge of two circles (up to isomorphism of covering spaces).