Math 444/539 Lecture 10

$S^1 \subseteq \mathbb{C}$

Thm: $\pi_1(S^1, 1) \cong \mathbb{Z}$ w/ generator $\gamma : I \to S^1$, $\gamma(t) = e^{2\pi i t}$

Define

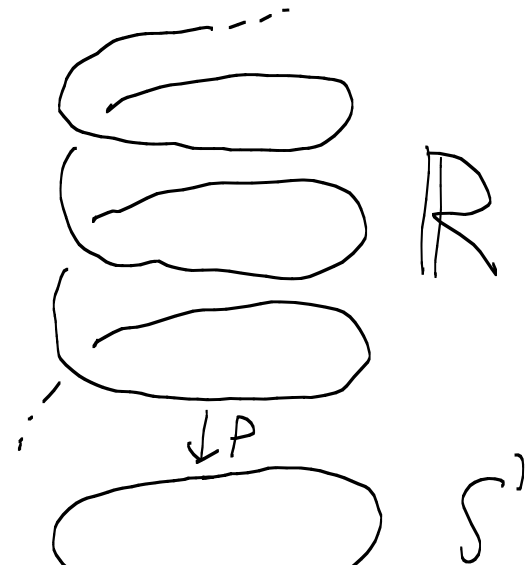
$\rho : \mathbb{R} \to S^1$

$\rho(x) = e^{2\pi i x}$

A *lift* to $\mathbb{R}$ of a map $f : X \to S^1$ is a map $\tilde{f} : X \to \mathbb{R}$ st diagram

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

commutes, ie st $f = p \circ \tilde{f}$.



Key Lemma

a) $f : I \to S^1$ path, $\tilde{x} \in p^{-1}(f(0))$
$\Longrightarrow \exists !$ lift $\tilde{f} : I \to \mathbb{R}$ w/ $\tilde{f}(0) = \tilde{x}$

b) $F : I \times I \to S^1$ map, $\tilde{f}_0 : I \to \mathbb{R}$ lift of $f_0 = F(\cdot, 0)$
$\Longrightarrow \exists$ lift $\tilde{F} : I \times I \to \mathbb{R}$ st $\tilde{F}(x, 0) = \tilde{f}_0(x)$

RmK: $\tilde{F}$ in b unique too, but we won't need this

Pf of Thm assuming key lemma

Define

$\gamma_n : I \to S^1$, $\quad \tilde{\gamma}_n : I \to \mathbb{R}$

$\gamma_n(t) = e^{2\pi i n t}$, $\quad \tilde{\gamma}_n(t) = nt$

so $\tilde{\gamma}_n$ lift of γ_n

Define

$$\psi : \mathbb{Z} \to \pi_1(S^1, 1)$$
$$\psi(n) = [\gamma_n] = [\underbrace{\gamma_1 \cdot \gamma_1 \cdot \cdots \cdot \gamma_1}_{n\ \gamma_1\text{'s}}]$$

Claim: ψ surjective

Consider 1-based loop $f: I \to S^1$. Let $\tilde{f}: I \to \mathbb{R}$ be lift w/ $\tilde{f}(0) = 0$.

$$p(\tilde{f}(1)) = 1 \implies \tilde{f}(1) = n \in \mathbb{Z}$$

Define

$$F: I \times I \to S^1$$
$$F(x,t) = p\left((1-t)\tilde{f}(x) + t\tilde{\gamma}_n(x)\right)$$

Then

$$F(x,0) = p(\tilde{f}(x)) = f(x)$$
$$F(x,1) = p(\tilde{\gamma}_n(x)) = \gamma_n(x)$$
$$F(0,t) = p((1-t)\tilde{f}(0) + t\tilde{\gamma}_n(0)) = p(0) = 1$$
$$F(1,t) = p((1-t)\tilde{f}(1) + t\tilde{\gamma}_n(1)) = p(n) = 1$$

$\therefore f \sim \gamma_n$, so $[f] \in \mathrm{Im}(\psi)$.

Claim: ψ injective

Asm $\psi(n) = \psi(m)$, so $\gamma_n \sim \gamma_m$

Let $F: I \times I \to S^1$ be homotopy from γ_n to γ_m

Let $\tilde{F}: I \times I \to \mathbb{R}$ be lift w/ $\tilde{F}(x,0) = \tilde{\gamma}_n(x)$

<u>Subclaim</u>: $\tilde{F}(0,t)=0$ and $\tilde{F}(1,t)=n$

$F(0,\cdot)$ is constant path 1, $\tilde{F}(0,\cdot)$ lift of $F(0,\cdot)$ starting at 0. Uniqueness of path lifting $\Longrightarrow$ $\tilde{F}(0,\cdot)$ is constant path 0.

Similarily, $\tilde{F}(1,\cdot)$ is constant path n.

Hence $\tilde{F}(\cdot,1)$ is lift of γ_m starting at 0
Uniqueness of path lifting $\Longrightarrow$ $\tilde{F}(x,1)=\tilde{\gamma}_m(x)$
$\therefore$ $n=\tilde{F}(1,1)=\tilde{\gamma}_m(1)=m$ ▨

For pf of Key lemma, need following result from point-set topology:

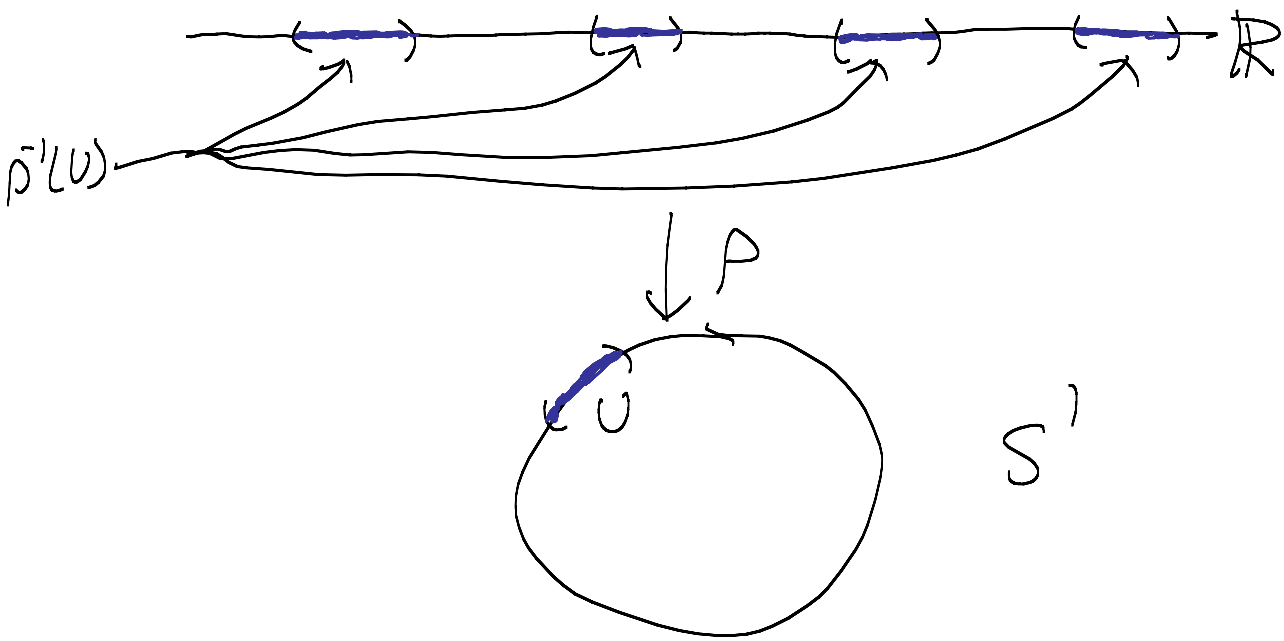
<u>Thm</u>: X compact metric space, $\{U_\alpha\}$ open cover of X
$\Longrightarrow$ $\{U_\alpha\}$ has <u>Lebesgue number</u> $\delta>0$:
$\forall x\in X$, $\exists\alpha$ st $B_\delta(x)\subseteq U_\alpha$

<u>Pf of Key Lemma</u>:

Give $S^1\subseteq\mathbb{C}$ induced metric.
Pick $\varepsilon>0$ small (eg $\varepsilon=0.1$)

Step 1: Lemma true if $f(I)$ (resp. $F(I\times I)$) lies in ε-ball

Pf's of parts a + b similar, will do part b.
Consider $F: I\times I \to S^1$ + $\tilde{f}_0: I \to \mathbb{R}$ as in lemma.
Let $U \subseteq S^1$ be ε-ball st $F(I\times I) \subseteq U$
Let $\tilde{U} \subseteq \mathbb{R}$ be cpt of $p^{-1}(U)$ w/ $\tilde{f}_0 \subseteq \tilde{U}$
Key observation: $p|_{\tilde{U}}: \tilde{U} \to U$ homeomorphism



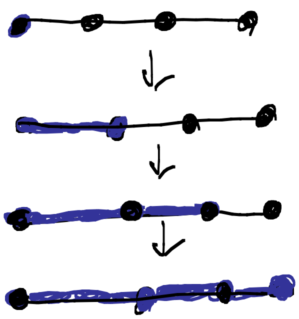
Hence $\tilde{F} = (p|_{\tilde{U}})^{-1} \circ F$ is desired lift, which is clearly unique.

Step 2: Part a, general case

Consider $f: I \to S^1$ and $\tilde{x} \in p^{-1}(f(0))$ as in part a

Claim: Can find $0=a_1<a_2<\cdots<a_k=1$ st $f([a_i,a_{i+1}]) \subseteq \varepsilon$-ball for $1\leq i<k$

Let $\{V_\alpha\}$ be cover of S^1 by ε-balls
$U_\alpha = f^{-1}(V_\alpha)$
δ = Lebesgue # of $\{U_\alpha\}$
$\Longrightarrow$ enough to choose a_i st $a_{i+1}-a_i<\delta$ for $1\leq i<k$



Lift f "1 segment at time":
Set $\tilde{f}(0)=\tilde{x}$
Asm $\tilde{f}$ defined on $[0,a_i]$
Step 1 $\Longrightarrow$ $\exists!$ lift of $f|_{[a_i,a_{i+1}]}$ starting at $\tilde{f}(a_i)$
$\therefore$ Can extend $\tilde{f}$ to $[0,a_{i+1}]$.
Uniqueness of $\tilde{f}$ follows from uniqueness in Step 1

Step 3: Part b, general case

Let $F: I\times I \to S^1$ and $\tilde{f}_0$ be as in Lemma

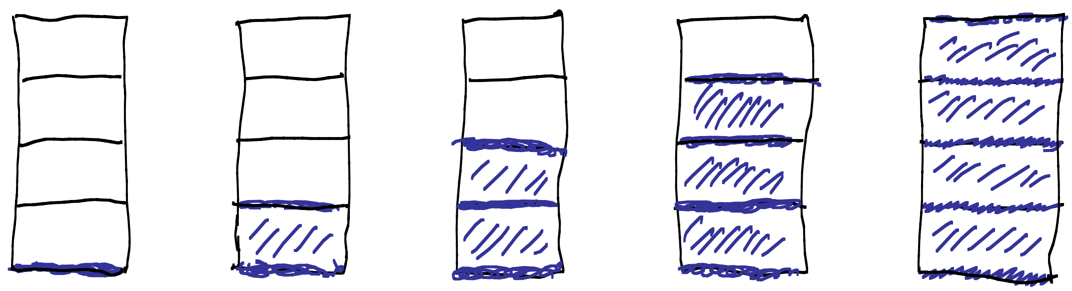
Claim: Can find $0=a_1<a_2<\cdots<a_k=1$ and $0=b_1<b_2<\cdots<b_\ell=1$ st $F([a_i,a_{i+1}]\times[b_j,b_{j+1}])$ contained in ε-ball

Lebesgue # argument like in Step 2

Set $F_i := F|_{[a_i, a_{i+1}] \times I}$ and $\tilde{f}_{0,i} = \tilde{f}_0|_{[a_i, a_{i+1}]}$

<u>Claim</u>: Can find lift $\tilde{F}_i : [a_i, a_{i+1}] \times I \longrightarrow \mathbb{R}$ of F_i st $\tilde{F}_i(\cdot, 0) = \tilde{f}_{0,i}$

Lift F_i "1 square at time"



Set $\tilde{F}_i(\cdot, 0) = \tilde{f}_{0,i}$
Asm $\tilde{F}_i$ defined on $[a_i, a_{i+1}] \times [0, b_j]$
Step 1 $\Longrightarrow \exists$ lift of F_i restricted to $[a_i, a_{i+1}] \times [b_j, b_{j+1}]$ "starting at $\tilde{F}(\cdot, b_j)$"
$\therefore$ Can extend $\tilde{F}_i$ to $[a_i, a_{i+1}] \times [0, b_{j+1}]$

<u>Claim</u>: $\tilde{F}_i(a_{i+1}, t) = \tilde{F}_{i+1}(a_{i+1}, t)$ for $t \in I$

Both $\tilde{F}_i(a_{i+1}, \cdot)$ and $\tilde{F}_{i+1}(a_{i+1}, \cdot)$ lifts of $F(a_{i+1}, \cdot)$ starting at $\tilde{f}_0(a_{i+1})$, so claim follows from uniqueness of path lifting (Step 2)

$\therefore$ Can "glue" together $\tilde{F}_i$ to get desired $\tilde{F}$.