Setup: \( X \) top space w/ open cover \( \{U_\alpha\} \)
\( \rho \in \bigcap U_\alpha \)
\( U_\alpha \cap U_\beta = U_\alpha \cap U_\beta \cap U_\gamma \) path-connected for all \( \alpha, \beta, \gamma \)
\( \varphi_{\alpha \beta}: \pi_1(U_\alpha \cap U_\beta, \rho) \to \pi_1(U_\rho, \rho) \) induced map
\( \Psi: \bigotimes_{\alpha} \pi_1(U_\alpha, \rho) \to \pi_1(X, \rho) \) map from univ. property of \( X \)

Thm (Seifert-van Kampen): \( \Psi \) surjective and \( \ker(\Psi) = R \), \( R \) normal subgroup gen by
\( \{(\varphi_{\alpha \beta}(\gamma))(\varphi_{\beta \alpha}(\gamma)) \mid \gamma \in \pi_1(U_\alpha \cap U_\beta, \rho), \alpha, \beta \) arbitrary? \)

Pf:
Already proved \( \Psi \) surjective, must prove \( \ker(\Psi) = R \)

Defn: A factorization of a \( p \)-based loop \( \gamma \) in \( X \) is expression
\( [\gamma] = [\gamma_1] \cdots [\gamma_k] \)
\( \) w/ \( \gamma_i \subseteq U_{\alpha_i} \) for some \( \alpha_1, \ldots, \alpha_k \)

Defn: Let \( [\gamma] = [\gamma_1] \cdots [\gamma_k] \) w/ \( \gamma_i \subseteq U_{\alpha_i} \) be factorization
a) A type I move: if for some \( i \) have \( \gamma_i \subseteq U_{\alpha_{i-1}} \), then replace \( U_{\alpha_i} \) w/ \( U_{\alpha_{i-1}} \).

b) A type II move: If \( U_{\alpha_i} = U_{\alpha_{i+1}} \), then change to
\( [\gamma_{i-1}] \cdots [\gamma_i \cdot \gamma_{i+1}] \cdots [\gamma_k] \)

2 factorizations of \( [\gamma] \) are equivalent if differ by sequence of type I/II moves or their inverses.

Key Claim: Any 2 factorizations of a \( p \)-based loop \( \gamma \) are equivalent.
Key Claim $\implies$ Ker$(\Psi) = R$:
If $[x_1] \cdots [x_k] \in$ Ker$(\Psi)$, then key claim says equivalent
to trivial loop. But type I moves correspond to
applying relns from $R$ and type II moves correspond
to applying relns in $\Psi$. Conclude: $[x_n] \cdots [x_k] \in R$.

Pf of Key Claim:

Asm
$[y] = [x_1] \cdots [x_k] + [y] = [x_1'] \cdots [x_k']$

2 factorizations. Set
$S = x_1 \cdots x_k + S' = x_1' \cdots x_k'$
$S \sim S'$ $\implies$ $\exists F : I \times I \to X$ s.t
$F(s,0) = S(s)$, $F(s,1) = S'(s)$,
$F(0,t) = F(1,t) = p$

Can decompose $I \times I$ into rectangles $R_1, \ldots, R_n$ s.t
$F(R_i) \subseteq U_{\beta_i}$ and every $p$ of $I \times I$ lies in $\leq 3$
rectangles:
Set $V_{\alpha} = F^{-1}(U_{\alpha})$ and let $\varepsilon > 0$ be Leb. # of
$\{V_{\alpha}\}$. Then need only choose $R_i$ w/ diam $< \varepsilon$
in following pattern:

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R_1 R_2 R_3 R_4
R_5 R_6 R_7 R_8
R_9
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Number like this

For vertex $v$ of tiling, choose path $\eta_v$ from $p$
to $F(v)$ in $\cap$ of the $U_{\alpha}$ that $F$ of the rectangles
Containing $p$ lie in (at most 3 $V_{\alpha}$'s).
For a path $\gamma$ in "grid" from LHS to RHS get factorization $F(\gamma)$ of $[\gamma]$:

Let $v_1, \ldots, v_m$ be vertices of $\gamma$ and let $e_i$ be edge in grid from $v_i$ to $v_{i+1}$. Then $F(v_i) = F(v_m)$ and have factorization

$$\left[ F(e_1) \cdot \bar{\eta}_{F(v_1)} \right] \cdot \left[ \eta_{F(v_2)} \cdot F(e_2) \cdot \bar{\eta}_{F(v_2)} \right] \cdot \left[ \eta_{F(v_3)} \cdot F(e_3) \cdot \bar{\eta}_{F(v_3)} \right] \cdots \left[ \eta_{F(v_{m-1})} \cdot F(e_{m-1}) \right]$$

For $0 \leq i \leq N$, let $\gamma_i$ be grid path separating $R_1, \ldots, R_i$ from $R_{i+1}, \ldots, R_N$:
Easy to check: 

a) $\mathcal{F}(v_0)$ equivalent to $\mathcal{F}(v_{i+1})$

b) $\mathcal{F}(v_i)$ equivalent to $[\bar{x}_i] - [\bar{x}_k]$

c) $\mathcal{F}(v_N)$ equivalent to $[\bar{x}_i] - [\bar{x}_e]$

Conclude: $[\bar{x}_i] - [\bar{x}_k]$ equivalent to $[\bar{x}_i] - [\bar{x}_e]$. [ ]