Thm: A tree $T$ is contractible to any $p \in V(T)$

pf:

\underline{Step 1: $T$ finite}

By induction on $|V(T)|$

- If $|V(T)| = 1$, trivial
- If $|V(T)| > 1 \Rightarrow$ Evolve $1$ vertex $v_0$

WLOG, $v_0 \neq p$

Let $e_v \in E(T)$ be adjacent to $v_0$

Can contract $T$ to $T' = T \setminus (\text{Int}(e_v) \cup \{v_0\})$

Induction $\Rightarrow$ Can contract $T'$ to $p$

\underline{Step 2: General $T$}

If $T$ connected $\Rightarrow \exists F^0: T^{(0)} \times I \to T$ s.t.

$F^0(x_o) = x$, $F^0(x_0) = p$,

$F^0(p_o, t) = p$

Must extend $F^0$ to $F: T \times I \to T$

Consider $e = e_v \in E(T)$ w/ end points $p_o, p_o \in T^{(0)}$

$e \cong I$, so $\exists (e \times I) = \text{square}$

Define

$F_e^0: (e \times I) \to T$

$F_e^0(x_o, t) = x$, $F_e^0(x_0, t) = p$

$F_e^0(p_o, t) = F^0(p_o, t)$

$F_e^0(p_o, t) = F^0(p_o, t)$

$F_e^0(e \times I)$ compact $\Rightarrow \exists$ finite tree $T^1 \subseteq T \wedge F_e^0(e \times I) \subseteq T^1$
Step 1 \( \Rightarrow \) \( T' \) 1-connected
\[ \mathcal{E}(e \in I) \cong S' \], so 1-connectivity implies can extend \( F_e : \mathcal{E}(e \in I) \to T \) to \( F_e : e \times I \to T \)
Define
\[ F : T \times I \to T \]
\[ F|_{e \times I} = F_e \]
By construction, this is well-defined continuous function with
\[ F(x, 0) = x, \ F(x, 1) = \rho, \ F(\rho, t) = \rho \]

**Thm:** \( X \) connected graph, \( p \in V(X) \)
\[ \Rightarrow \Pi_1(X, p) \) free grp

pf:
\[ T \subseteq X \) maximal tree
\[ \{e_{\alpha}\} \) edges of \( X \) not in \( T \)

Divide \( e_{\alpha} \) into 3 segments:

Set \( f_{\alpha} = e_{\alpha'} U e_{\alpha''} \)
So \( f_{\alpha} \) open in \( e_{\alpha} \)
Define
\[ G_\alpha = T \cup e_\alpha \]
\[ U_\alpha = T \cup e_\alpha \cup \left( \bigcup_{\beta} U_\beta \right) \]

**Facts:**

a) \( U_\alpha \) open
b) \( U_\alpha \) def. retracts onto \( G_\alpha \)
c) \( \alpha \neq \beta \Rightarrow U_\alpha \cap U_\beta = T \cup (U_\beta \delta) \), which def retracts onto \( T \)

\[ \Rightarrow U_\alpha \cap U_\beta \text{ path-connected and } \]
\[ \pi_1(U_\alpha \cap U_\beta, p) = 1 \]

\[ U_\alpha \cap U_\beta \cap U_\gamma \text{ path-connected} \]

\[ \therefore \text{Can apply Seifert-van Kampen to } \bigcup U_\alpha, \]

and by (x) the "relations" \( R \) are trivial

\[ \Rightarrow \pi_1(X, p) = \bigast_{\alpha} \pi_1(U_\alpha, p) \]

\[ \cong \bigast_{\alpha} \pi_1(G_\alpha, p) \]

Hence enough to prove:

**Claim:** \( \pi_1(G_\alpha, p) \cong \mathbb{Z} \)

\( p_1, p_2 \) endpts of \( e_\alpha \)

\( \varepsilon \) injective path in \( T \) from \( p_1 \) to \( p_2 \)

Clear: Each cpt of \( G_\alpha \setminus (\varepsilon \cup e_\alpha) \) is tree, so \( G_\alpha \)
def. retracts to \( e_\alpha \cong S^1 \)
Algorithm for finding $\pi_1(X,p)$ for graph $X$

1) Find max tree $T \subseteq X$
2) Let $\{e_\alpha\}$ be edges not in $T$
3) Orient $e_\alpha$ and let $i_\alpha$ and $t_\alpha$ be its initial and terminal vertices
4) For $v \in V(X)$, let $\delta_v$ be unique injective path in $T$ from $p$ to $v$

Conclusion: $\pi_1(X,p)$ is free grp w/ generators $\{X_\alpha\}$ w/ $X_\alpha = \delta_{i_\alpha}^{-1} e_\alpha \delta_{t_\alpha}$ loop based at $p$

Ex:

$\pi_1(X,p)$ is free grp on $X_1, \ldots, X_5$, where:

- $X_1 = ae_1f$
- $X_2 = be_2d$
- $X_3 = be_3f$
- $X_4 = fg e_4$
- $X_5 = fe_5f$

Rmk: Basis for $\pi_1(X,p)$ not canonical 'cause it depends on $T$

However, # of generators indep. of $T$