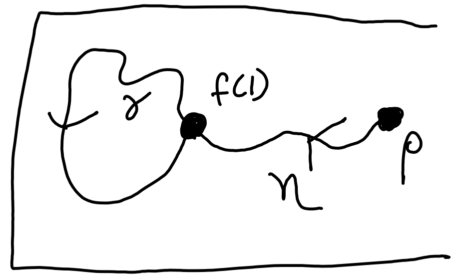
Math 444/539 Lecture 17

Last lecture: Determined $\pi_1(X,p)$ for X a graph

Goal today: Extend to higher dim CW-cpx's

Informally: ⓐ $X^{(1)}$ determines generators
ⓑ 2-cells give relations
ⓒ K-cells for $K \geq 3$ don't affect π_1

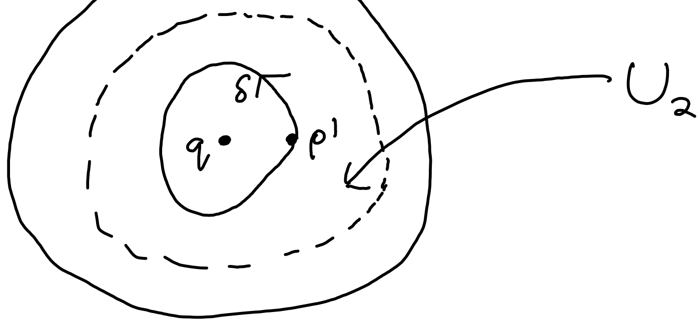
Thm (Attaching 2-cell): X space, $p \in X$, $f: S^1 \to X$ map, $Y = X \sqcup D^2/\sim$ w/ $v \in \partial D^2 \sim f(v) \in X$.
Let
η = path in X from p to $f(1)$
$\gamma': I \to S^1$, $\gamma'(t) = e^{2\pi i t}$
$\gamma = f \circ \gamma'$



$\Longrightarrow \pi_1(Y,p) \cong \pi_1(X,p)/N$, w/ N normal subgrp generated by $[\eta \cdot \gamma \cdot \overline{\eta}]$

Restatement: If $\pi_1(X,p) \cong \langle S \mid R \rangle$ and w expression for $\eta \cdot \gamma \cdot \eta'$ in S, then $\pi_1(Y,p) \cong \langle S \mid R \cup \{w\} \rangle$

pf of thm:
Let $p', q \in D^2$, $U_2 \subseteq D^2$, and $\delta: I \to D^2$ be:



Set $U_1 = Y \setminus q$. Then

a) $\pi_1(U_2, p') = 1$

b) $U_1 \cap U_2 = U_2 \setminus q$ path-connected

c) $\pi_1(U_1 \cap U_2, p') \cong \mathbb{Z}$ w/ gen $[\delta]$

Seifert-van Kampen $\Longrightarrow$

$$\pi_1(Y, p') \cong \pi_1(U_1, p') * \pi_1(U_2, p')/R$$
$$= \pi_1(U_1, p')/R$$

with R normal sub gen by $[\delta]$.

Change basept to $f(1)$:

$$\pi_1(Y, f(1)) \cong \pi_1(U_1, f(1))/R'$$

with R' normal subgrp gen by $[\gamma]$

Change basept to p:

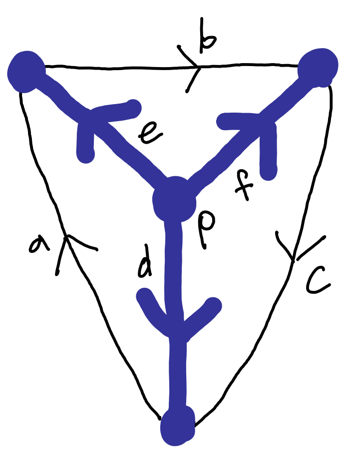
$$\pi_1(Y, p) \cong \pi_1(U_1, p)/N$$

with N normal subgrp gen by $[\eta \cdot \gamma \cdot \overline{\eta}]$

U_1 def. retracts to X, so $\pi_1(U_1, p) \cong \pi_1(X, p)$ ☐

Iterating thm, can calc $\pi_1(X, p)$ for any 2d CW cpx X

Ex: $X =$



w/ D^2 attached to abc.

Using indicated max tree,
$\pi_1(X^{(1)}, p) \cong$ free grp w/ gen
$x_1 = da\overline{e}$, $x_2 = eb\overline{f}$, $x_3 = fc\overline{d}$

Relation is $dabc$. In terms of x_1, x_2, x_3,
$dabc = x_1 x_2 x_3$

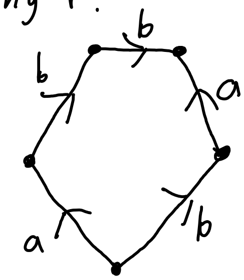
$\Rightarrow \pi_1(X, p) = \langle x_1, x_2, x_3 \mid x_1 x_2 x_3 \rangle$

<u>Thm</u>: For any grp G, there exists a 2d CW cpx X w/ $\pi_1(X,p) \cong G$. If G finitely presentable, then X can be chosen to be compact.

pf:
Write $G = \langle S \mid R \rangle$. Define $X^{(1)} = \bigvee_{\alpha \in S} S^1$ w/ wedge point p, so $\pi_1(X^{(1)}, p) \cong \langle S \mid \rangle$.

For each $r \in R$, attach 2-cell as follows:
Write $r = s_1^{e_1} \cdots s_k^{e_k}$ w/ $s_i \in S$, $e_i = \pm 1$
Divide up ∂D^2 into k segments labeled and oriented using r.

<u>Ex</u>:



for $r = abba^{-1}b^{-1}$

Attach D^2 so that edge labeled s_i wraps around loop s_i in appropriate direction

Let X = resulting cpx

Above thm $\Rightarrow \pi_1(X, p) = \langle S \mid R \rangle$ ▨

<u>Thm</u> (Attaching k-cells, $k>2$): X space, $p\in X$, $f: S^{k-1} \to X$ map,
$Y = X \sqcup D^k/\sim$ w/ $v \in \partial D^k \sim f(v) \in X$.
$\Longrightarrow \pi_1(Y,p) \cong \pi_1(X,p)$ if $k>2$.

pf:
Pick $q, p' \in \mathrm{Int}(D^k)$. Set $U_1 = Y \setminus q$ and let $U_2 \subseteq \mathrm{Int}(D^k)$ be open ball w/ $q, p' \in U_2$. Then

a) $U_1 \cap U_2 \cong U_2 \setminus q$. Since $U_2 \cong \mathring{D}^k$ and $k>2$, get that $U_1 \cap U_2$ is path-connected and $\pi_1(U_1 \cap U_2, p') = 1$

b) $\pi_1(U_2, p') = 1$

$SvK \Longrightarrow \pi_1(Y, p') \cong \pi_1(U_1, p')$.

U_1 def. retracts onto X, so conclude that $\pi_1(Y,p) \cong \pi_1(X,p)$.

$\boxed{///}$

<u>Cor</u>: X CW-cpx, $p \in X^{(2)}$. $\Longrightarrow \pi_1(X,p) \cong \pi_1(X^{(2)}, p)$.

<u>Ex</u>: $\mathbb{R}P^n$ has CW-cpx structure s.t. k-skeleton ($k \leq n$) is $\mathbb{R}P^k$.
$\Longrightarrow \pi_1(\mathbb{R}P^n, p) \cong \pi_1(\mathbb{R}P^2, p) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$.