Last lecture: Determined $\pi_1(X, p)$ for $X$ a graph

Goal today: Extend to higher dim CW-complexes

Informally:
1. $X^0$ determines generators
2. 2-cells give relations
3. $k$-cells for $k \geq 3$ don't affect $\pi_1$

**Thm (Attaching 2-cell):** $X$ space, $p \in X$, $f: S^1 \to X$ map,

Let

$Y = X \cup D^2/\sim$ w/ $v \sim f(v) \in X$.

Let

$\gamma = \text{path in } X \text{ from } p \text{ to } f(1)$

$\gamma': I \to S^1$, $\gamma'(t) = e^{2\pi i t}$

$\gamma = f \circ \gamma'$

$\Rightarrow \pi_1(Y, p) \cong \pi_1(X, p)/N$, w/ $N$ normal subgroup generated by $[\gamma \cdot \gamma' \cdot \gamma^{-1}]$

Restatement: If $\pi_1(X, p) = \langle S \mid R \rangle$ and $w$ expression for $\gamma \cdot \gamma' \cdot \gamma^{-1}$ in $S$, then $\pi_1(Y, p) = \langle S \mid R \cup \{w\} \rangle$

**pf of thm:**

Let $p, q \in D^2$, $U_2 \subseteq D^2$, and $S: I \to D^2$ be:
Set $U_i = Y \setminus q_i$. Then

a) $\pi_1(U_2, p') \cong 1$

b) $U_i \cap U_2 = U_2 \setminus q_i$ path-connected

c) $\pi_1(U, U_i \cap U_2, p') \cong \mathbb{Z}$ w/ gen [8]

Seifert-van Kampen

$\pi_1(Y, p') \cong \pi_1(U_i, p') \ast \pi_1(U_2, p') / R$

$= \pi_1(U_i, p') / R$

with $R$ normal subgen by [8].

Change basept to $f(c_i)$:

$\pi_1(Y, f(c_i)) \cong \pi_1(U_i, f(c_i)) / R'$

with $R'$ normal subgrp gen by [8]

Change basept to $p$:

$\pi_1(Y, p) \cong \pi_1(U_i, p) / N$

with $N$ normal subgrp gen by $[N : Y : N]$.

$U_i$ def. retracts to $X$, so $\pi_1(U_i, p) \cong \pi_1(X, p)$ □

Iterating thm, can calc $\pi_1(X, p)$ for any 2d CW $c_{p \times X}$

Ex: $X = \begin{array}{c}
\text{with } \Box^3 \text{ attached to } abc
\end{array}$

Using indicated max tree,

$\pi_1(X^{(p)}) \cong \text{free grp w/ gen }$

$x_1 = dae$, $x_2 = eb\overline{f}$, $x_3 = fca$.
Relation is \( dabc \). In terms of \( x_1, x_2, x_3 \),
\[
dabc = x_1x_2x_3
\]
\[\implies \pi_1(X, p) = \langle x_1, x_2, x_3 \mid x_1x_2x_3 \rangle\]

**Thm:** For any group \( G \), there exists a 2d CW complex \( X \) with \( \pi_1(X, p) \cong G \). If \( G \) finitely presentable, then \( X \) can be chosen to be compact.

**pf:** Write \( G = \langle S \mid R \rangle \). Define \( X^\infty = \bigvee_{s \in S} S^1 \) with wedge point \( p \), so \( \pi_1(X^\infty, p) \cong \langle S \rangle \).

For each \( r \in R \), attach 2-cell as follows:
- Write \( r = s_1^{e_1} \cdots s_k^{e_k} \) with \( s_i \in S, e_i = \pm 1 \)
- Divide up \( \partial D^2 \) into \( k \) segments labeled and oriented using \( r \).

![Example diagram](attachment:image.png) for \( r = aba^{-1}b^{-1} \)

Attach \( D^2 \) so that edge labeled \( s_i \) wraps around loop \( s_i \) in appropriate direction.

Let \( X = \) resulting complex

Above thm \( \implies \pi_1(X, p) = \langle S \mid R \rangle \) \( \blacksquare \)
**Thm:** (Attaching $k$-cells, $K > 2$): $X$ space, $p \in X$, $f: S^{k-1} \to X$ map, $\ Y = X \cup D^k / \sim$ w/ $v \in \partial D^k \sim f(v) \in X$.  
$\Rightarrow \pi_i(Y, p) \cong \pi_i(X, p)$ if $K > 2$.

**pf:**
Pick $q, p' \in \text{Int}(D^k)$. Set $V_1 = Y \setminus q$ and let $U_2 \subseteq \text{Int}(D^k)$ be open ball w/ $q, p' \in U_2$. Then

1) $U_1 \cap U_2 \cong U_2 \setminus q$. Since $U_2 \cong D^k$ and $K > 2$, get that $U_1 \cap U_2$ is path-connected and $\pi_1(U_1 \cap U_2, q) = 1$

2) $\pi_1(U_2, p') = 1$

$S^n K \Rightarrow \pi_i(Y, p') \cong \pi_i(U_1, p')$.

$U_1$ def. retracts onto $X$ so conclude that $\pi_i(Y, p) \cong \pi_i(X, p)$.

**Cor:** $X$ CW-cplx, $p \in X^{(n)} \Rightarrow \pi_i(X, p) \cong \pi_i(X^{(n)}, p)$.

**Ex:** $\mathbb{R}P^n$ has CW-cplx structure s/t $K$-skeleton $(KSH)$ is $\mathbb{R}P^n$.

$\Rightarrow \pi_i(\mathbb{R}P^n, p) \cong \pi_i(\mathbb{R}P^n, p) \cong \mathbb{Z}/2\mathbb{Z}$ for $n > 2$. 