Def'n: A **knot** is an embedding $f: S^1 \to \mathbb{R}^3$

Ex: a) unknot

b) trefoil

**Problematic Example:** wild knot

Def'n: A **tame knot** is a knot $f: S^1 \to \mathbb{R}^3$ s.t. exists finitely many pts $x_1, \ldots, x_n \in S^1$ s.t. for each cpt $C$ of $S^1 \setminus \{x_1, \ldots, x_n\}$, $f(C)$ is straight line
Remark: Will still draw tame knots as smooth curves. You should imagine that they are divided into so many straight segments that from a distance they appear smooth.

Equivalence of Knots
Want to say that 2 knots are equivalent if you can move one to the other in space, as if they were lengths of rope w/ ends joined.

Formal Def'ly: Let \( f, g: S^1 \to \mathbb{R}^3 \) be knots

a) Attempt # 1: \( f \) and \( g \) are homotopic if
\[ \exists F: S^1 \times I \to \mathbb{R}^3 \quad s.t. \quad F(x,0) = f(x) \quad and \quad F(x,1) = g(x) \]

Problem: Can pass strands through each other, so all knots are homotopic

\[ \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \]

b) Attempt # 2: \( f \) and \( g \) are isotopic if \( \exists F: S^1 \times I \to \mathbb{R}^3 \)
\[ s.t. \quad F(x,0) = f(x), \quad F(x,1) = g(x) \]
and maps \( f_t: S^1 \to \mathbb{R}^3 \) w/ \( f_t(x) = F(x,t) \) are embeddings for all \( t \)

Problem: Can "shrink" knotted region through embeddings until it disappears, so all tame knots are isotopic
C) Attempt $\#3$ : if $f, g$ are ambient isotopic if 
\[ \exists H : \mathbb{R}^3 \times I \to \mathbb{R}^3 \text{ s.t. maps } h_t : \mathbb{R}^3 \to \mathbb{R}^3 \] 
\[ h_t(x) = H(x, t) \text{ are homeomorphisms for all } t, \quad h_0 = \text{id}, \] 
and $g = h_1 \circ f$

"Move space along w/ knot"

Rmk: The maps $f_t = h_t \circ f : S^1 \to \mathbb{R}^3$ are embeddings for all $t$, so ambient isotopic knots are isotopic

Will say knots $f, g : S^1 \to \mathbb{R}^3$ are equivalent if they are ambient isotopic

Rmk: Above wild knot not equivalent to any tame knot

Main Problem of Knot Theory: Find invariants to distinguish non-equivalent knots.
Will confuse knot \( f: S^1 \rightarrow \mathbb{R}^3 \) with its image \( f(S^1) \).

**Observation:** If knots \( K_1 \) and \( K_2 \) are equivalent, then \( \mathbb{R}^3 \setminus K_1 \cong \mathbb{R}^3 \setminus K_2 \).

\( \mathbb{R}^3 \setminus K \) knows a lot about \( K \):

**Thm (Gordon-Luecke):** \( K_1, K_2 \subset \mathbb{R}^3 \) tame knots

\[ K_1 \text{ equivalent to } K_2 \iff \mathbb{R}^3 \setminus K_1 \cong \mathbb{R}^3 \setminus K_2 \]

or \( K_2 \) w/ orientation reversed.

**Defn:** The group of a knot \( K \) is \( \pi_1(\mathbb{R}^3 \setminus K) \).

Often useful to view knot \( K \subset \mathbb{R}^3 \) as living in \( S^3 = \mathbb{R}^3 \cup \{\infty\} \).

**Lemma:** \( K \subset \mathbb{R}^3 \text{ knot } \implies \pi_1(\mathbb{R}^3 \setminus K) \cong \pi_1(S^3 \setminus K) \).

Ref:

\( U_1 = \mathbb{R}^3 \setminus K \), \( U_2 \subset S^3 \setminus K \) small open ball around \( \infty \).

Then

\[ S^3 \setminus K = U_1 \cup U_2 \]

\[ U_1 \cap U_2 = U_2 \setminus \{\infty\} \text{ path-connected} \]

\[ \pi_1(U_2) = \pi_1(U_1 \cup U_2) = 1 \]

\[ S^3 \setminus K \implies \pi_1(S^3 \setminus K) \cong \pi_1(U_1) \]

Recall from HW 1: \( S^3 = X_1 \cup X_2 \) w/ \( X_0 = D^2 \times S^1 \) and \( X_0 \cup X_2 \cong T^3 \) embedded in std way.
Theorem: K unknot. Then \( \pi_1(\mathbb{S}^3 \setminus K) \cong \mathbb{Z} \).

Proof:
Let \( \mathbb{S}^3 = X_1 \cup X_2 \) be as above. Then \( \mathbb{S}^3 \setminus K \) deformation retracts to \( X_2 \), so \( \pi_1(\mathbb{S}^3 \setminus K) \cong \pi_1(X_2) \cong \mathbb{Z} \).