Math 444/539, Lecture 3

Def'n: An <u>n-manifold</u> is a 2^{nd} countable Hausdorff space X s.t. ~~[illegible]~~ $\forall p \in X$, $\exists$ nbhd U of p w/

chart

$$U \cong \mathring{D}^n \cong \{\vec{x} \in \mathbb{R}^n \mid \sum x_i^2 < 1\}.$$

Ex: a) $\mathbb{R}^n$

b) S^n. Consider $x \in S^n$. WLOG, $x_{n+1} > 0$.

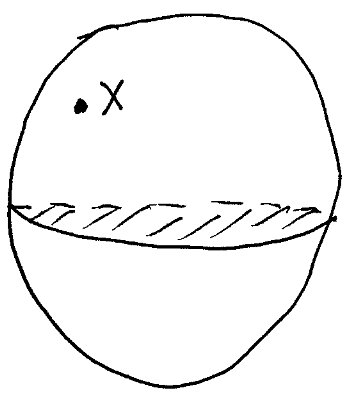
Have chart

$$U = \{\vec{y} \in S^n \mid y_{n+1} > 0\}$$

w/ homeo

$$\varphi : U \to \mathring{D}^n$$

$$\varphi(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n)$$



c) M n-mfld, $V \subseteq M$ open

$\Longrightarrow V$ n-mfld

d) M m-mnfld, N n-mnfld $\Longrightarrow$ $M \times N$ $(m+n)$-mnfld

$(p,q) \in M \times N$

$U \subseteq M$ chart for p

$V \subseteq M$ chart for q

$\Longrightarrow U \times V$ nbhd of (p,q) w/

$U \times V \cong \mathring{D}^m \times \mathring{D}^n \cong \mathring{D}^{m+n}$

<u>Hard Thm</u>: For $n \neq 4$, all n-mnflds can be given structure of CW-cpx

<u>Rmk</u>: Open for $n=4$.

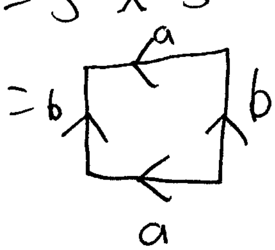
<u>Goal</u>: Classify cpt 2-mnflds (surfaces)

<u>3 Key Examples</u>

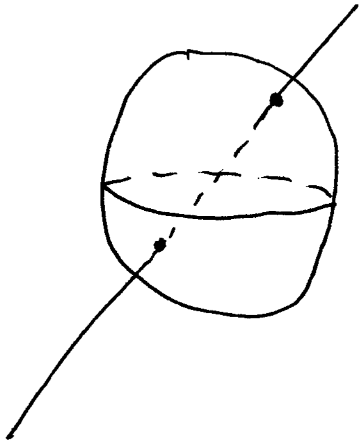
a) S^2

b) $T^2 =$ 

$= S^1 \times S^1$

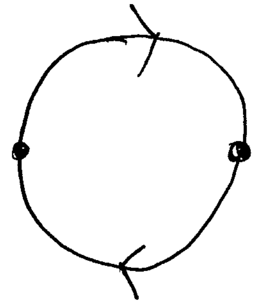
$=$ 

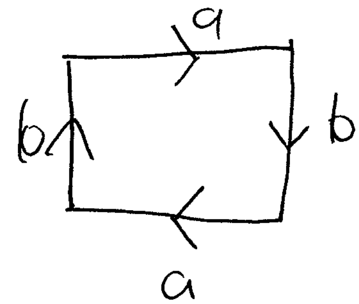
c) $\mathbb{RP}^2$ = "lines in $\mathbb{R}^3$ through 0"



$= S^2 / x \sim -x$

$= D^2 / {}_{x \in \partial D^2,\ x \sim -x}$

$=$ 

$=$ 

Connected Sum

S_1, S_2 surfaces

$D_i \subseteq S_i$ closed discs (ie $D_i \cong D^2$)

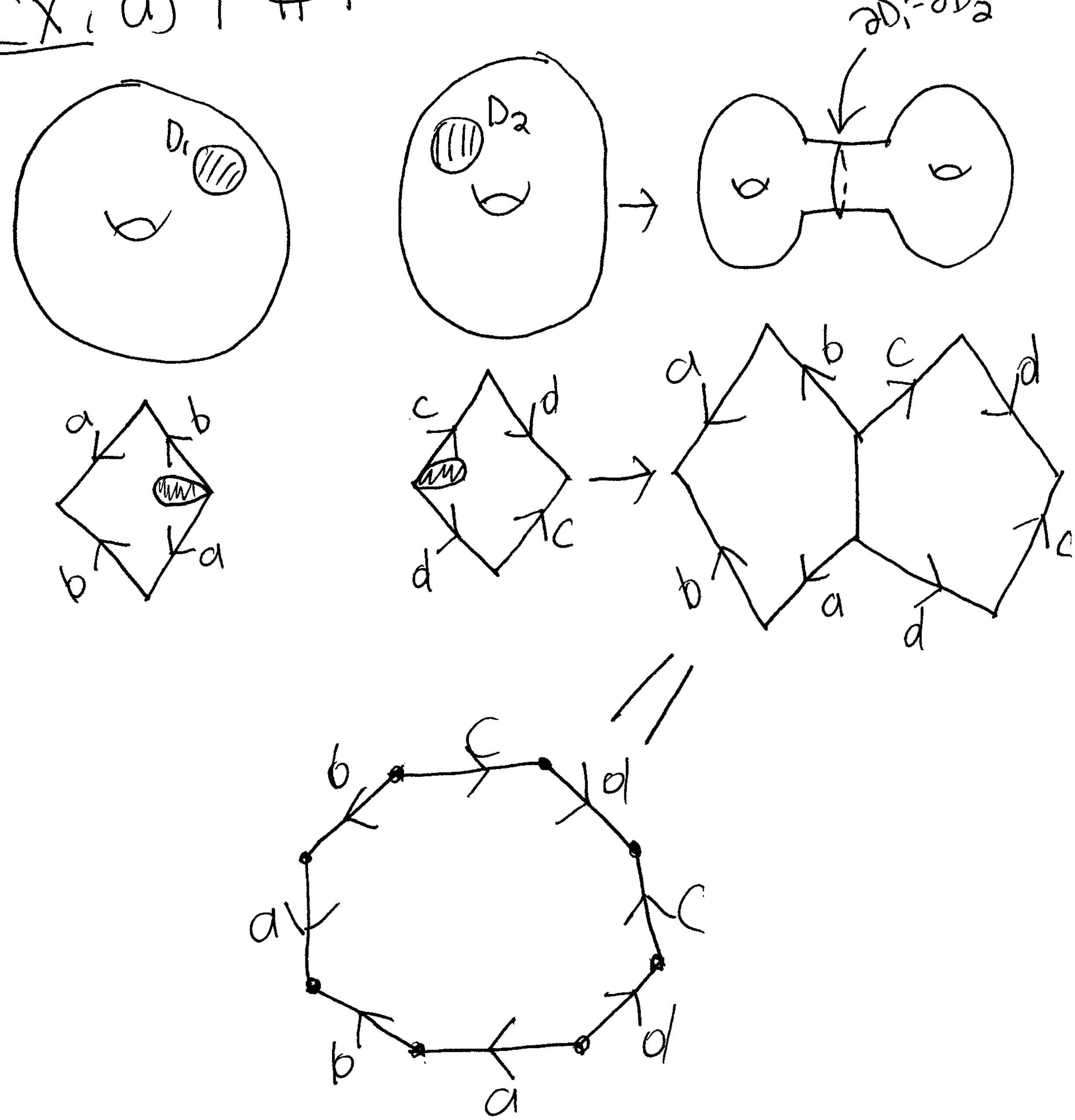
$h : \underset{\cong S^1}{\partial D_1} \to \underset{\cong S^1}{\partial D_2}$ homeo

The connected sum of $S_1 + S_2$ is

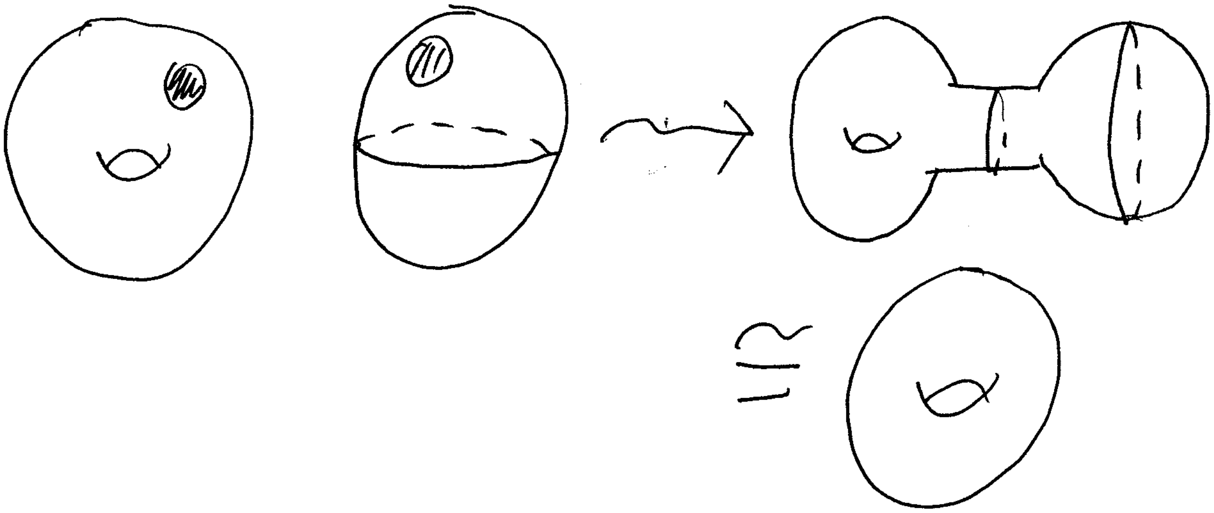
$$S_1 \# S_2 = \Big((S_1 \setminus Int(D_1)) \sqcup (S_2 \setminus Int(D_2)) \Big) \Big/ \begin{array}{l} x \sim h(x) \\ \text{for } x \in \partial D_1 \end{array}$$

Thm: $S_1 \# S_2$ is a well-defined (ie independent of D_i's + h) surface.

Ex: a) $T^2 \# T^2$



b) $T^2 \# S^2 \cong T^2$



In fact,

<u>Lemma</u>: $\Sigma \# S^2 \cong \Sigma$ for all Σ.

c) $\underbrace{T^2 \# \cdots \# T^2}_{g\ T^2\text{'s}}$ = "genus g surface"

=

g handles

<u>Rmk</u>: a) $S_1 \# S_2 \cong S_2 \# S_1$

b) $S_1 \# (S_2 \# S_3) \cong (S_1 \# S_2) \# S_3$

<u>Upshot</u>: compact surfaces form commutative monoid under # w/ unit S^2

Structure of surface monoid

Thm 1: Σ cpt 2-mnfld

$$\Rightarrow \Sigma \cong \underbrace{(T^2 \# \cdots \# T^2)}_{g\ T^2\text{'s}} \# \underbrace{(\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2)}_{h\ \mathbb{R}P^2\text{'s}}$$

Rmk: $g = h = 0 \Rightarrow \Sigma \cong S^2$

Thm 2: $T^2 \# \mathbb{R}P^2 \cong \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$

Cor to thm 1, 2: Σ cpt 2-mnfld

$\Rightarrow$ either

$$\Sigma \cong T^2 \# \cdots \# T^2$$

or

$$\Sigma \cong \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2$$

Thm 3: The surfaces $T^2 \# \cdots \# T^2$ and $\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2$ are all distinct.

Will prove thm's 1 & 2 now

Thm 3 needs fund. grp