Math 444/539 Lecture 5

Goal: Study 1-d CW cpx's (ie graphs).

Defn: X graph.

a) 0-cells = vertices

$V(X) = \{\text{vertices of } X\}$

b) 1-cells = edges

$E(X) = \{\text{edges of } X\}$

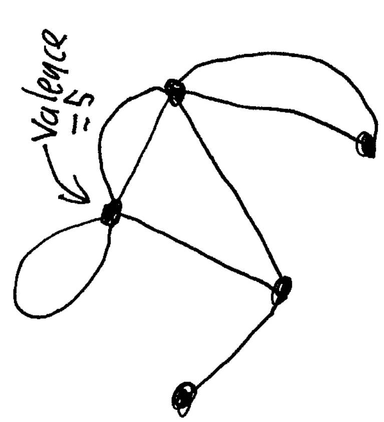
c) $e \in E(X) \Rightarrow e \cap V(X) = \{v, v'\}$ (possibly $v = v'$).

Say e joins v + v'

d) $v \in V(X)$

The valence of v is # of edges joining v to other vertices (counted 2x if they join v to itself)

Ex:



Rmk Can have ∞ many edges/vertices.

Def'n: X graph.

An <u>edge-path</u> in X is sequence $v_1, \ldots, v_k$ of vertices & $e_1, \ldots, e_{k-1}$ of edges s.t. e_i joins v_i & v_{i+1} for $1 \le i < k$.

Write

$$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} \cdots \xrightarrow{e_{k-1}} v_k$$

w/ e_i sometimes omitted.

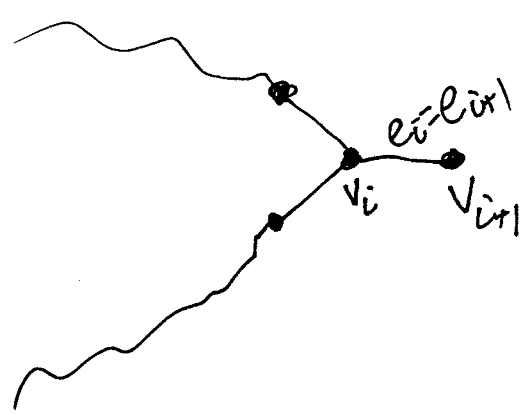
Easy Lemma: X graph

X path-connected $\iff$ $\forall\, v, w \in V(X)$, $\exists$ edge-path joining v & w.

Def'n: An edge-path

$$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} \cdots \xrightarrow{e_{k-1}} v_k$$

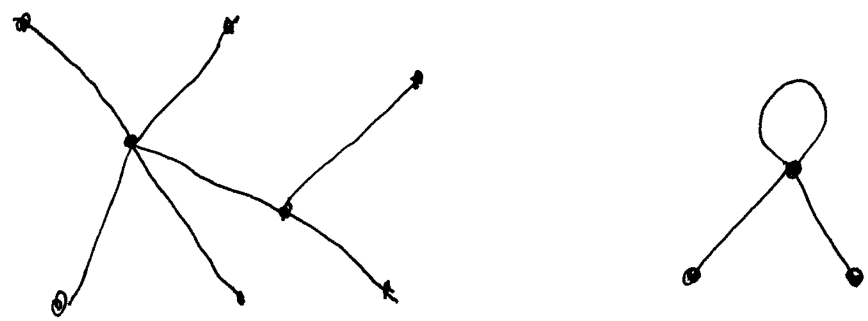
<u>backtracks</u> if $e_i = e_{i+1}$ and $v_i \neq v_{i+1}$ for some $1 \le i < k$.



Def'n: A <u>cycle</u> is an edge-path $v_1 - v_2 - \cdots - v_k$ that doesn't backtrack w/ $v_1 = v_k$ and $k > 1$.

Def'n: A <u>tree</u> is a connected graph w/ no cycles

Ex:



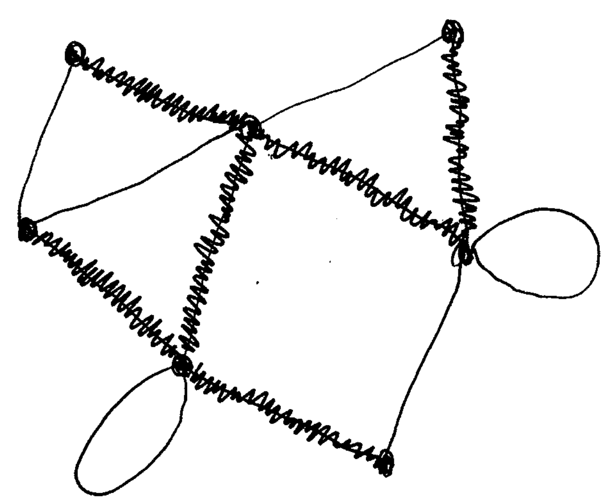
tree

not tree

Def'n: X graph, $Y \subseteq X$ subgraph.

Y is a <u>maximal tree</u> in X if Y is a tree and $V(Y) = V(X)$.

Ex:



<u>Thm</u>: Every connected graph X has a maximal tree.

pf:

Let $\mathcal{T} = \{ T \subseteq X \mid T \text{ a tree} \}$.

$\mathcal{T}$ partially ordered by inclusion.

<u>Goal</u>: Apply Zorn's lemma

Check conditions:

Let $\Theta \subseteq \mathcal{T}$ be totally ordered

Set $S = \bigcup_{T \in \Theta} T$

<u>Claim</u>: S connected

$v, w \in V(S)$

$\exists A \in \Theta$ w/ $v, w \in V(A)$ (since Θ totally ordered)

A connected

$\therefore S$ connected

<u>Claim</u>: S has no cycles

Asm S has cycle $v_1 \overset{e_1}{—} v_2 \overset{e_2}{—} \cdots \overset{e_{k-1}}{—} v_k$

$\exists B \in \Theta$ s.t. $v_i \in V(B)$ + $e_i \in E(B)$ for all i (since Θ totally ordered)

But B is a tree, contradiction.

$\therefore S$ is a tree, so $S \in \mathcal{T}$ is upper bd for Θ

Zorn $\Rightarrow$ $\mathcal{T}$ has max elt T.

Claim: $T \subseteq X$ is maximal tree

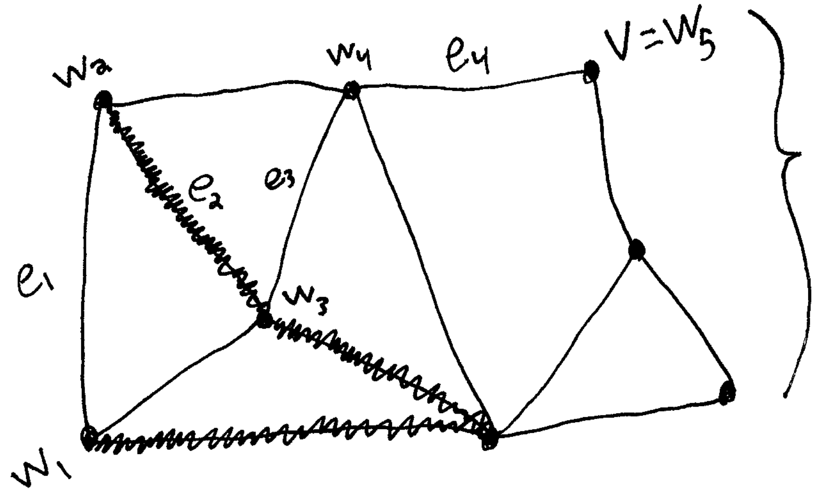
Asm not, so $\exists v \in V(X) \setminus V(T)$

X connected $\Rightarrow \exists$ edge path $w_1 \xrightarrow{e_1} \cdots \xrightarrow{e_{k-1}} w_k$

w/ $w_1 \in V(T)$, $w_k = v$.

$\exists 1 \leq i < k$ s.t. $w_i \in T$ but $w_{i+1} \notin T$

$\Rightarrow T \cup e_i$ a tree, contradiction.



In above pf, could take $i = 3$

$\square$

Recall: G graph $\Rightarrow \chi(G) = |V(G)| - |E(G)|$.

Thm: G finite connected graph.

$\Rightarrow$ a) $\chi(G) \leq 1$

b) $\chi(G) = 1 \Leftrightarrow G$ a tree pf:

Step 1: T finite tree $\Longrightarrow \chi(T)=1$

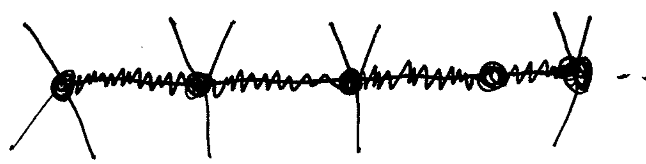
By induction on $|V(T)|$.

$|V(T)|=1 \Rightarrow T=\bullet \Longrightarrow \chi(T)=1-0=1.$

Asm $|V(T)|>1$ and claim true for trees T' w/ $|V(T')|<|V(T)|$.

Claim: Can find valence 1 vertex $v_0 \in V(T)$

Otherwise, can find ∞-length edge path: start w/ some $w_1 \in V(T)$, keep continuing along edges you haven't yet visited



Let $e_0 \in E(T)$ be adjacent to v_0.

Set $T' = T \setminus \{Int(e_0) \cup v_0\}$

T' tree, so induction $\Longrightarrow \chi(T')=1$

$\Longrightarrow \chi(T) = \chi(T')+1-1 = \chi(T')=1$

Step 2: G not tree $\Longrightarrow \chi(G) < 1$

$T \subseteq G$ maximal tree.

T maximal $\Longrightarrow |V(T)| = |V(G)|$

G not tree $\Longrightarrow |E(T)| < |E(G)|$

$$\chi(G) = |V(G)| - |E(G)|$$
$$= |V(T)| - |E(G)| + |V(T)| - |V(T)|$$
$$= \chi(T) - |E(G)| + |V(T)|$$
$$< \chi(T) = 1$$

$\boxtimes$