Math 444/539 Lecture 6

Goal: Prove

Thm: Σ cpt surface $\Longrightarrow$ Σ is connected sum of T^2's and $\mathbb{RP}^2$'s.

Will give pf due to Zeeman

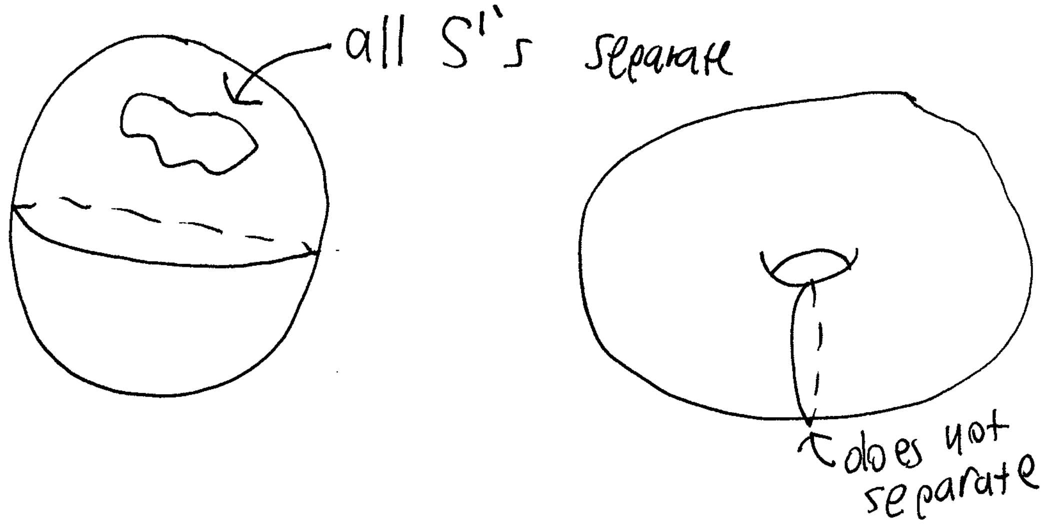
Key Lemma: Σ cpt surface

$\Longrightarrow$ a) $\chi(\Sigma) \leq 2$

b) $\chi(\Sigma) = 2 \Longrightarrow \Sigma \cong S^2$ (2d Poincaré conjecture)

c) $\chi(\Sigma) < 2 \Longrightarrow \exists$ embedding $f: S^1 \hookrightarrow \Sigma$ s.t. $\Sigma \setminus f(S^1)$ is connected.

Rmk: Conclusion c false for S^2 by Jordan curve thm



Assume Key lemma true

Proof of Thm

By "backwards induction" on $\chi(\Sigma)$

Key lemma $\Longrightarrow \chi(\Sigma) \leq 2$ & thm true if $\chi(\Sigma)=2$

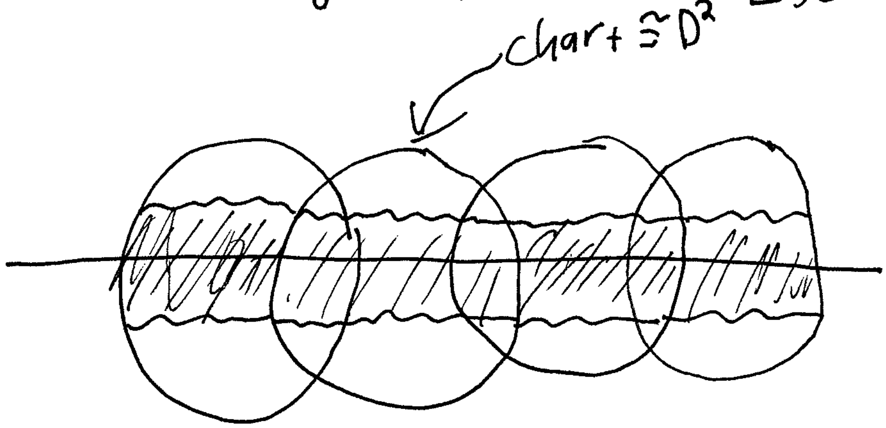
Asm $\chi(\Sigma)<2$ & thm true for Σ' w/ $\chi(\Sigma')>\chi(\Sigma)$

Key lemma $\Longrightarrow \exists$ embedding $f: S^1 \hookrightarrow \Sigma$ w/ $\Sigma \setminus f(S^1)$ connected

Set $\gamma = f(S^1)$.

Let N = "thickened up" γ ← tubular nbhd

Construction: N locally $[0,1] \times [-1,1]$ w/ γ = "center line" $[0,1] \times \{0\}$.



Construct N chart by chart

Eventually N comes back and joins up w/ itself.

$\Longrightarrow N \cong [a,b]\times[-1,1]/\sim$ where $\sim$ identifies the sides $\{a\}\times[-1,1]$ + $\{b\}\times[-1,1]$

$\gamma \subseteq N$ and $\gamma \cap N = [a,b]\times\{0\}/\sim$

<u>2 cases</u>

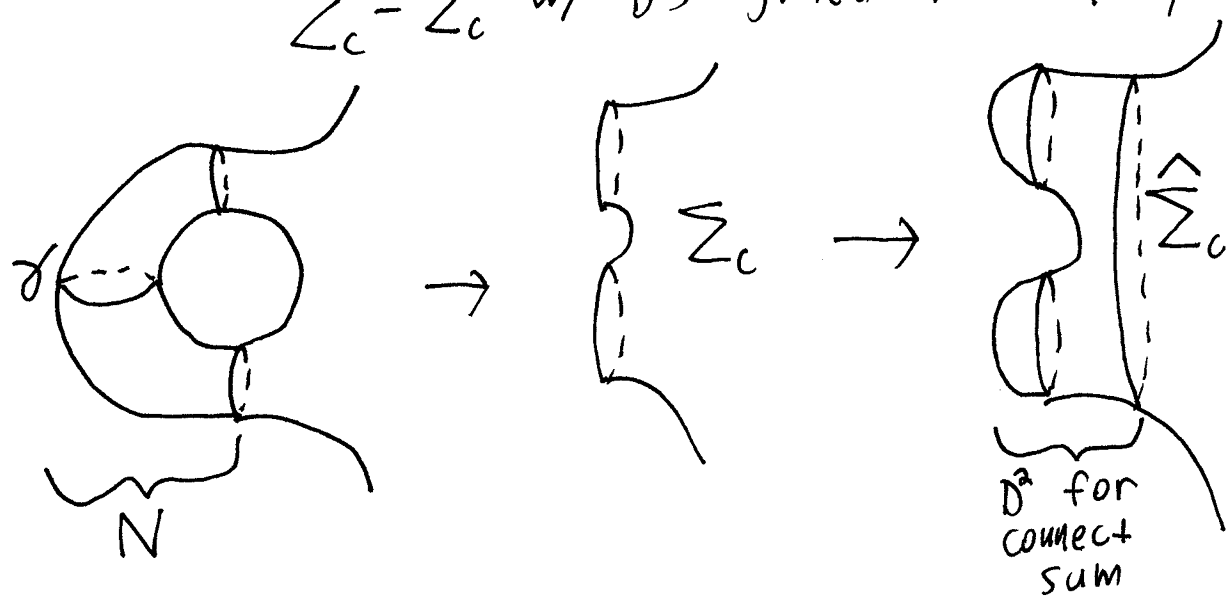
a) N glued w/o twist, so $N \cong S^1 \times [-1,1]$ (annulus)

b) N glued w/ twist, so $N \cong$ Möbius band

<u>Case a:</u> N annulus

Set $\Sigma_c = \Sigma \setminus Int(N)$ ($\cong S^1 \times (-1,1)$)

$\hat{\Sigma}_c = \Sigma_c$ w/ D^2's glued to 2 bdry S^1's.



Key Observation: $\Sigma \cong \hat{\Sigma}_c \# T^2$

$$\Rightarrow \chi(\Sigma) = \chi(\hat{\Sigma}_c) + \chi(T^2) - 2$$
$$= \chi(\hat{\Sigma}_c) - 2,$$
so $\chi(\hat{\Sigma}_c) > \chi(\Sigma)$

Induction $\Rightarrow$ $\hat{\Sigma}_c$ is connect sum of T^2's and $\mathbb{RP}^2$'s

$\Rightarrow$ so is Σ.

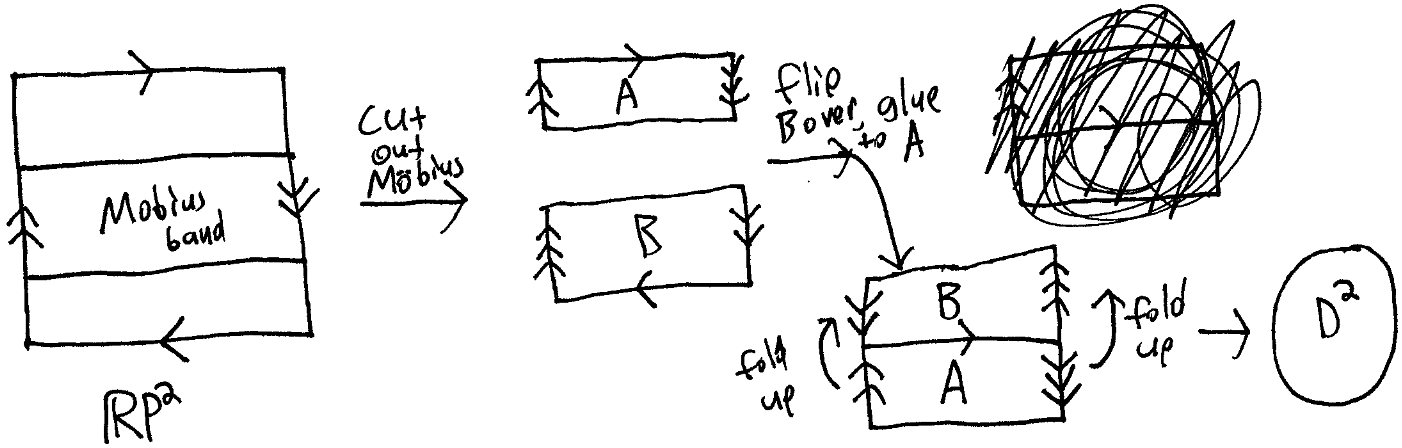
<u>Case b</u>: N Möbius band

Like in case a, can cut and cap to get $\hat{\Sigma}_c$, but now $\Sigma \cong \hat{\Sigma}_c \# \mathbb{RP}^2$ ← see below for why

Again done by induction ▨

<u>We used</u>: S surface $\Rightarrow$ $S \# \mathbb{RP}^2$ same as cutting out disc from S + gluing in Möbius band.

equiv., $\mathbb{RP}^2 \setminus$ disc = Mobius band



Recall:

<u>Key Lemma</u>: Σ cpt surface

$\Longrightarrow$ a) $\chi(\Sigma) \leq 2$

b) $\chi(\Sigma) = 2 \Longrightarrow \Sigma \cong S^2$

c) $\chi(\Sigma) < 2 \Longrightarrow \exists$ embedding $f: S^1 \hookrightarrow \Sigma$ st $\Sigma \setminus f(S^1)$ connected.

pf:

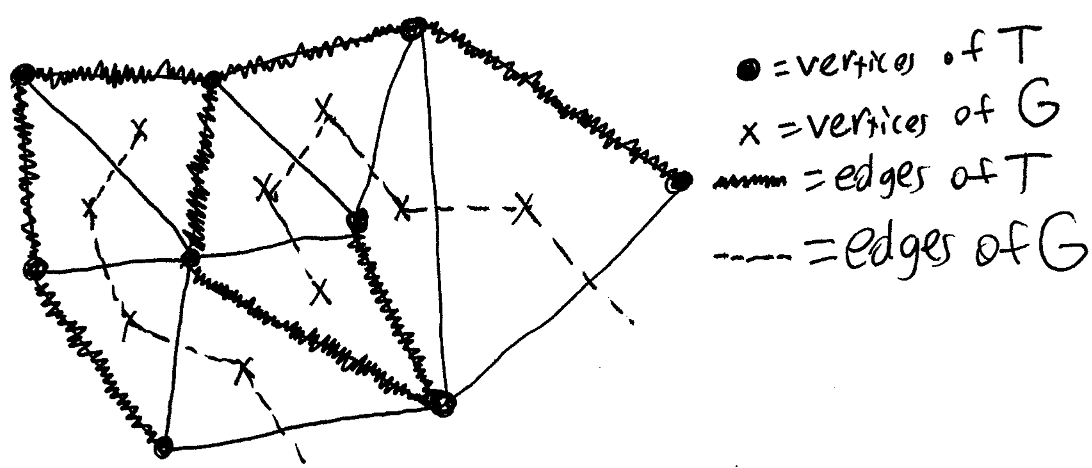
Choose triangulation for Σ

T = maximal tree in $\Sigma^{(1)}$

Let G = "dual graph" to T:

vertices = 2-cells of Σ

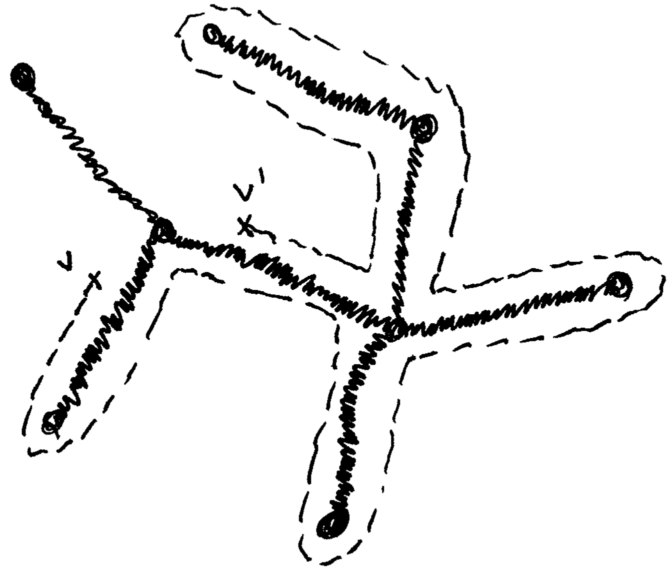
edges: S_1 joined to S_2 if $S_1 \cap S_2$ = edge not in T



<u>Claim</u>: G connected.

$v, v' \in V(G)$

Can connect v + v' by following path "around" T:



Let

$v = \#$ vertices of Σ

$e = \#$ edges of Σ

$f = \#$ faces of Σ

<u>Observe</u>: $v = |V(T)|$, $e = |E(T)| + |E(G)|$, $f = |V(G)|$

$$\Rightarrow \chi(\Sigma) = v - e + f = |V(T)| - (|E(T)| + |E(G)|) + |V(G)|$$

$$= \chi(T) + \chi(G) = 1 + \chi(G)$$

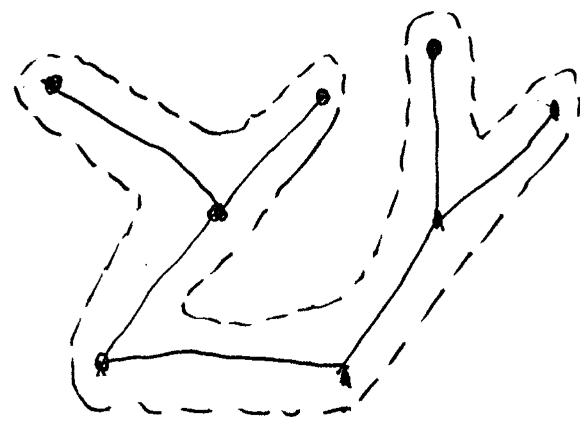
$\chi(G) \leq 1 \Rightarrow \chi(\Sigma) \leq 2 \leftarrow$ conclusion a.

2 cases

a) $\chi(\Sigma)=2$

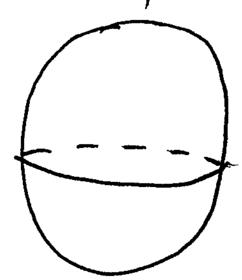
$\Rightarrow \chi(G)=1$, so G a tree

small nbhds of T, $G \cong D^2$



Σ = these 2 D^2's glued together along bdry

$\cong S^2$



b) $\chi(\Sigma)<2$

$\Rightarrow \chi(G)<1$, so G not tree

$\Rightarrow \exists$ embedding $f: S^1 \hookrightarrow G \subseteq \Sigma$

Claim: $\Sigma \setminus f(S')$ connected

Consider $p, q \in \Sigma \setminus f(S')$

Can find paths in $\Sigma \setminus f(S')$ from p to vertex, and similarily for q

In $T \subseteq \Sigma \setminus f(S')$, can find paths between any 2 vertices

$\therefore$ Can find path in $\Sigma \setminus f(S')$ from p to q, so $\Sigma \setminus f(S')$ Connected.