Goals
a) Prove $T^3 \# \mathbb{RP}^3 \cong \mathbb{RP}^3 \# \mathbb{RP}^3 \# \mathbb{RP}^3$

b) Classify surfaces w/ bdry

Defn: The Klein bottle is $K^2 = \text{[diagram]}$

Picture: Can't draw $K^2$ in $\mathbb{R}^3$ w/o self-intersections. Need $\mathbb{R}^4$.

Lemma: $K^2 \cong \mathbb{RP}^3 \# \mathbb{RP}^3$

pf: Recall that $\mathbb{RP}^2 = \text{[diagram]}$ and $\mathbb{RP}^2 \setminus \text{disc} \cong \text{[diagram]}$. Must use 4th dim to avoid self-intersections here.
Hence
\[ \mathbb{R}P^2 \# \mathbb{R}P^2 = \mathbb{R}P^2 \]
Goal: a thus follows from:

Thm: $T^3 \# \mathbb{RP}^3 \cong K^3 \# \mathbb{RP}^3$

pf:

Observe:

$I.$ $T^3 / \text{disc} \cong$

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$I.$ $K^2 / \text{disc} \cong$

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$I.$ $\mathbb{RP}^3 / \text{disc} \cong$

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$I.$ Hence

$T^3 \# \mathbb{RP}^3 \cong$

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$\quad \quad \quad \quad \quad \quad \quad \quad$ \quad \quad \quad \quad \quad \quad \quad \quad$ Central Möbius band

$\quad \quad \quad \quad \quad \quad \quad \quad$ \quad \quad \quad \quad \quad \quad \quad \quad$ left side of "handle"
Drag left side of "handle" around central Möbius band get that this is homeo. to

\[ \cong K^2 \# \mathbb{RP}^2 \]

Since Möbius band has "twist", \( \Box \)

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**Surfaces w/ Bdry**

**Defn:** An \( n \)-manifold w/ boundary is a \( 2 \)-nd countable Hausdorff space \( X \) s.t. for all \( p \in X \), there exists a nbhd \( U \) of \( p \) s.t. one of the following holds:

a) \( U \cong \mathbb{D}^n = \{ x \in \mathbb{R}^n \mid \sum x_i^2 < 1 \} \)

b) \( U \cong \{ x \in \mathbb{R}^n \mid \sum x_i^2 < 1 \text{ and } x_n > 0 \} \); call this latter set \( \mathbb{B}^n_+ \).
**Vocabulary:** Let $X$ be an $n$-manifold with boundary.

a) $p \in X$ is an interior point if $p$ has a neighborhood $U$ with $U \cong \mathbb{B}^n$.

Define

$\text{Int}(X) = \{ p \in X \mid p$ interior pt $\}$.

Remark: This is different from point-set topology definition of interior.

b) $p \in X$ is a boundary point if $p$ is not interior pt.

Define

$\partial X = \{ p \in X \mid p$ boundary pt $\}$.

**Remarks:**

a) A manifold is a manifold with boundary $X$ such that $\partial X = \emptyset$; conversely, if $X$ is a manifold with boundary and $\partial X \neq \emptyset$, then $X$ is not a manifold.

b) If $p \in \partial X$, then $p$ has a neighborhood $U$ that is homeomorphic to $\partial \mathbb{B}^n$, i.e., $\psi : U \to \partial \mathbb{B}^n$.

It's true (but annoying to prove) that conversely, if such a $\psi$ exists, then $p$ is not an interior pt. In particular, the point $\partial \mathbb{B}^n$ has no neighborhood $V$ with $V \cong \mathbb{B}^n$.

![Diagram of a half-disc and point](image)
Ex: a) $D^n$ is n-mfld w/ boundary.
   $\text{Int}(D^n) = \partial D^n$
   $\partial D^n \cong S^{n-1}$

b) $\Sigma$ cpt surface, $B \subseteq X$ subspace w/ $B = \partial B^3$.
   Then $\Sigma \setminus \text{Int}(B)$ is an mfld w/ boundary.

c) Can also remove multiple discs

**Lemma:** $X$ mfld w/ boundary $\Rightarrow \exists x$ is $(n-1)$-mfld.

pf:
Let $\psi : U \rightarrow D^n$ be chart w/ $\psi(p) = 0$.
Then a pt $\partial x \in D^n$ is image of bdry pt iff $x_n = 0$.

pt w/ nonzero last coord. has nbhd $\cong \mathbb{R}^n$
pt w/ nonzero last coord. has nbhd $\cong B^n_+$. 
\[ \varphi \text{ maps } \bigcup \mathcal{X} \text{ homeo. onto } \\{ x \in \mathbb{R}^n \mid x \cdot e_0 = 0 \} \cong \mathbb{B}^{n-1} \]
\[ \Rightarrow \bigcup \mathcal{X} \cong \mathbb{B}^{n-1} \text{ is chart for } p \circ \mathcal{X} \]

**Cor.** \( \Sigma \) cpt surface w/ boundary

\[ \Rightarrow \partial \Sigma \text{ disjoint union of finite # of } S^1 \text{'s.} \]

**pf:**

\( \exists X \) compact 1-manifold \[ \square \]

**Thm:** \( \Sigma_1, \Sigma_2 \) cpt surfaces w/ bdry

\[ \Sigma_1 \cong \Sigma_2 \iff \]

\[ \begin{align*}
& a) \text{ both orientable or not orientable} \\
& b) \chi(\Sigma_1) = \chi(\Sigma_2) \\
& c) \text{ both have same # of bdry cpts.}
\end{align*} \]

**pf:**

\[ \Rightarrow \text{ trivial} \]

\[ \iff \text{ Let } \hat{\Sigma}_i \text{ be } \Sigma_i \text{ w/ discs glued to all bdry cpts.} \]

\[ \hat{\Sigma}_i \text{ cpt surface (without bdry)} \]

Claim: \( \chi(\hat{\Sigma}_i) = \chi(\Sigma_i) + n, \) where \( n \) is # of boundary cpts.

Triangulate \( \Sigma_i \)

Then \( \hat{\Sigma}_i \) obtained by adding 3-cell glued to each bdry cpt, so \( \chi(\hat{\Sigma}_i) = \chi(\Sigma_i) + n. \)
Conclude: $\chi(\hat{\Sigma}_1) = \chi(\hat{\Sigma}_2)$.

Since $\hat{\Sigma}_1 + \hat{\Sigma}_2$ either both orientable or both not orientable, classification of cpt surfaces $\Rightarrow \exists$ homeo. $\Psi: \hat{\Sigma}_1 \rightarrow \hat{\Sigma}_2$

Need following annoying lemma, whose proof is omitted:

**Lemma:** $S$ cpt surface

$B_1, \ldots, B_n \subseteq S$ disjoint subsets w/ $B_i \cong D^2$

$B'_1, \ldots, B'_n \subseteq S$ disjoint subsets w/ $B'_i \cong D^2$

$\Rightarrow \exists$ homeo $\Psi: S \rightarrow S$ w/ $\Psi(B_i) = B'_i$ for $1 \leq i \leq n$.

**Lemma** $\Rightarrow$ we can assume that $\Psi$ takes discs glued to bdry cpts of $\Sigma_1$ to discs glued to bdry cpts of $\Sigma_2$.

Hence $\Psi|_{\Sigma_1}$ is homeo from $\Sigma_1$ to $\Sigma_2$.