Math 444/539 Lecture 8

X = topological space

$I = [0,1]$

Def'n: a) A path from $p \in X$ to $q \in X$ is a continuous fcn $f: [0,1] \to X$ w/ $f(0) = p$, $f(1) = q$.

b) $f, g: [0,1] \to X$ paths from p to q. f equivalent to g (written $f \sim g$) if $\exists\, F: [0,1] \times [0,1] \to X$ w/

$$F|_{0 \times [0,1]} = p$$

$$F|_{1 \times [0,1]} = q$$

$$F|_{[0,1] \times 0} = f$$

$$F|_{[0,1] \times 1} = g$$

Easy: $\sim$ an equiv. rel; Will sometimes write $[f]$ for eq. class of path f.

Lemma: $f: I \to X$ path from p to q
$\varphi: I \to I$ fcn w/ $\varphi(0) = 0$ + $\varphi(1) = 1$
$\Longrightarrow f \sim f \circ \varphi$.

pf:

Define

$$F : I \times I \to X$$

$$F(t,s) = f\big((1-s)t + s\varphi(t)\big)$$

Check:

$$F(0,s) = f(s\varphi(0)) = f(0) = p$$

$$F(1,s) = f((1-s) + s\varphi(1)) = f(1) = q$$

$$F(t,0) = f(t)$$

$$F(t,1) = f(\varphi(t))$$

<u>Def'n</u>: f path from p to q

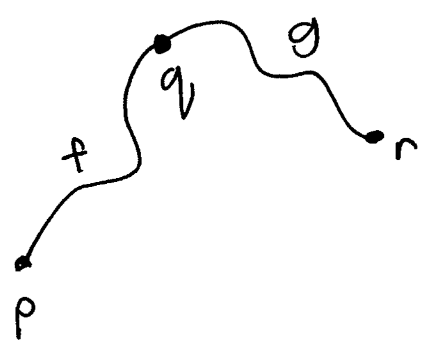
g path from q to r

Define $f \cdot g$ to be path

$$f \cdot g : I \to X$$

$$f \cdot g(t) = \begin{cases} f(2t) & 0 \le t \le 1/2 \\ g(2t-1) & 1/2 \le t \le 1 \end{cases}$$

from p to r.



<u>Lemma</u>: f_1, f_2 paths from p to q w/ $f_1 \sim f_2$

g_1, g_2 paths from q to r w/ $g_1 \sim g_2$

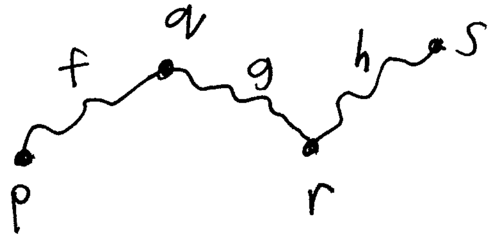
$\Longrightarrow [f_1 \cdot g_1] = [f_2 \cdot g_2]$

pf:

Easy

Mult. of eq. classes of paths thus well-defined.

<u>Lemma</u>: f path from p to q
g path from q to r
h path from r to s



$$\Longrightarrow [(f\cdot g)\cdot h] = [f\cdot (g\cdot h)]$$

pf:

$$(f\cdot g)\cdot h\,(t) = \begin{cases} f(4t) & 0 \leq t \leq 1/4 \\ g(4t-1) & 1/4 \leq t \leq 1/2 \\ h(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

$$f\cdot (g\cdot h)(t) = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(4t-2) & 1/2 \leq t \leq 3/4 \\ h(4t-3) & 3/4 \leq t \leq 1 \end{cases}$$

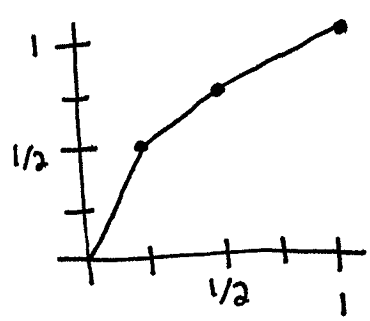
$$\Longrightarrow \big(f\cdot(g\cdot h)\big) = \big((f\cdot g)\cdot h\big)\circ \varphi \quad \text{w/}$$

$$\varphi : I \to I$$

$$\varphi(t) = \begin{cases} 2t & 0 \leq t \leq 1/4 \\ t + 1/4 & 1/4 \leq t \leq 1/2 \\ t/2 + 1/2 & 1/2 \leq t \leq 1 \end{cases}$$

$\boxed{//}$

<u>Rmk</u>: Graph of φ:



Def'n: For $p \in X$, let e_p be constant path

$$e_p(t) = p$$

from p to p.

Lemma: f path from p to q

$$\Longrightarrow [e_p \cdot f] = [f] = [f \cdot e_q]$$

pf:

$$e_p \cdot f(t) = \begin{cases} p & 0 \le t \le 1/2 \\ f(2t-1) & 1/2 \le t \le 1 \end{cases}$$

$\Longrightarrow e_p \cdot f = f \circ \varphi$ w/

$$\varphi : I \to I$$

$$\varphi(t) = \begin{cases} 0 & 0 \le t \le 1/2 \\ 2t-1 & 1/2 \le t \le 1 \end{cases}$$

$\Longrightarrow [e_p \cdot f] = [f]$

Similarily, $[f \cdot e_q] = [f]$. $\square$

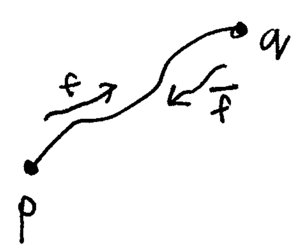
Def'n: f path from p to q

Define $\overline{f}$ to be path

$$\overline{f} : I \to X$$

$$\overline{f}(t) = f(-t+1)$$

from q to p.



Lemma: f path from p to q

$$\Longrightarrow [f \cdot \overline{f}] = [e_p] \text{ and } [\overline{f} \cdot f] = [e_q]$$

pf:

Since $\overline{\overline{f}} = f$, enough to prove $[f \cdot \overline{f}] = [e_p]$

Define

$$F : I \times I \longrightarrow X$$

$$F(t,s) = \begin{cases} f(2t) & 0 \le t \le s/2 \\ f(s) & s/2 \le t \le 1 - s/2 \\ f(2-2t) & 1 - s/2 \le t \le 1 \end{cases}$$

$F(\cdot, s)$ goes along f from p to $f(s)$, waits a while, then returns to p along $\overline{f}$

<u>Check</u>:

$$F(t,0) = f(0) = p$$

$$F(t,1) = (f \cdot \overline{f})(t)$$

$$F(0,s) = f(0) = p$$

$$F(1,s) = f(2-2) = p \quad \square$$

<u>Def'n</u>: A path is a <u>loop</u> / <u>closed</u> if its endpoints are equal.

<u>Def'n</u>: $p \in X$. The <u>fundamental group</u> of X w/ basepoint p is

$$\pi_1(X,p) = \{ [f] \mid f \text{ a loop at } p \}.$$

Above proves:

<u>Thm</u>: $\pi_1(X,p)$ is a group.

Ex: $p \in \mathbb{R}^n$. Then $\pi_1(\mathbb{R}^n, p) = 1$.

f loop based at p.

Define

$$F : I \times I \to \mathbb{R}^n$$

$$F(t,s) = sp + (1-s)f(t)$$

Then

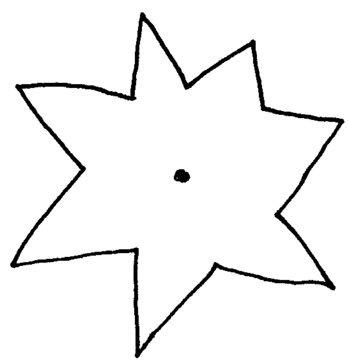
$$F(t,0) = f(t)$$

$$F(t,1) = p$$

$$F(0,s) = F(1,s) = p.$$

More generally,

Def'n: $U \subseteq \mathbb{R}^n$ is star-shaped relative to $p \in U$ if for all $x \in U$, the line seg. from p to x is in U



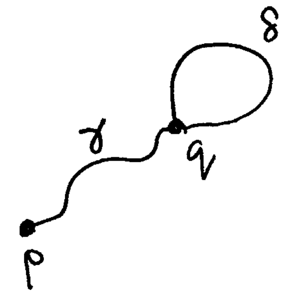
Lemma: $U \subseteq \mathbb{R}^n$ star-shaped relative to $p \in U$ $\Longrightarrow \pi_1(U, p) = 1$.

<u>Def'n</u>: γ eq class of paths from p to q.

Define

$$\varphi_\gamma : \pi_1(X,q) \to \pi_1(X,p)$$

$$\varphi_\gamma(\delta) = \gamma \cdot \delta \cdot \overline{\gamma}$$



<u>Lemma</u>: φ_γ a homomorphism

pf:

$$\begin{aligned}\varphi_\gamma(\delta_1 \cdot \delta_2) &= \gamma \cdot \delta_1 \cdot \delta_2 \cdot \overline{\gamma} \\ &= \gamma \cdot \delta_1 \cdot \overline{\gamma} \cdot \gamma \cdot \delta_2 \cdot \overline{\gamma} \\ &= \varphi_\gamma(\delta_1) \cdot \varphi_\gamma(\delta_2) \quad \blacksquare\end{aligned}$$

<u>Lemma</u>: $\varphi_\gamma \circ \varphi_{\overline{\gamma}} = 1$

pf:

$$\begin{aligned}\varphi_\gamma(\varphi_{\overline{\gamma}}(\delta)) &= \gamma \cdot \overline{\gamma} \cdot \delta \cdot \overline{\overline{\gamma}} \cdot \overline{\gamma} \\ &= \delta \quad \blacksquare\end{aligned}$$

<u>Cor</u>: φ_γ an isomorphism. Hence if $p, q \in X$ in same path component, then $\pi_1(X,p) \cong \pi_1(X,q)$

<u>Rmk</u>: This isomorphism depends on γ and is thus unnatural. Moral: don't ignore the basepoint!