Math 444/539 : Geometric Topology
Midterm

This exam is pledged. It consists of four problems, two of which are divided into two parts. Each part of each problem counts equally (in other words, each part of each problem counts for 1/6 of your grade). You are allowed five hours which begins once you start reading the problems on the next page. During the exam, you may take one 30 minute break during which you are not to read, write, or talk about anything math-related (these 30 minutes do not count towards your five hours). You are allowed to use Massey's "Algebraic Topology : An Introduction", your course notes, and the notes I posted on the webpage. No other sources (including internet sources) or consultation with other people are allowed.

Please write your solutions on a separate sheet of paper and staple the exam pages (including this front page) and your solution pages together.

Name :

Time Started :

Time break begins :

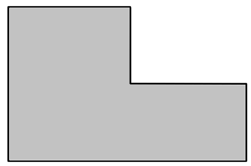
Time break ends :

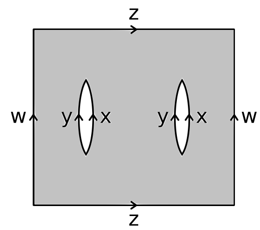
Time Ended :

Write the honor pledge :

Problems :

1. Consider the following two shapes:





   (a) Let X be the result of quotienting the left hand shape (the "sideways L") by the equivalence relation that identifies the boundary with itself in the following way. Consider a point p on the boundary of the shape. Let ℓ be a line segment of the boundary containing p. There are two line segments ℓ' and ℓ'' of the boundary that are parallel to ℓ. Let δ be the line through p which is perpendicular to ℓ. Then identify p with all points in $\delta \cap (\ell' \cup \ell'')$. Observe that except for points which are identified with corner points, each boundary point is identified with a unique other point on the boundary (corner points are identified with several other points).

   Identify the homeomorphism type of the surface X.

   (b) Let Y be the result of making the indicated identifications to the right hand shape (a square with two slits cut out). Identify the homeomorphism type of the surface Y.

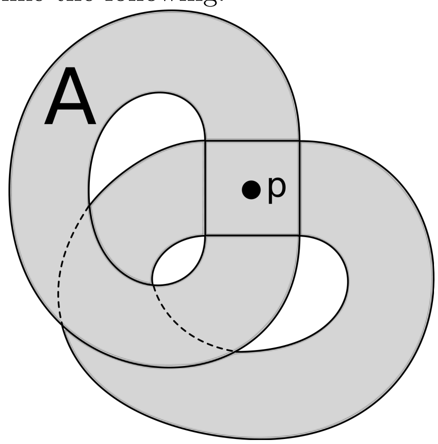
2. Let X be a space. The *suspension* of X, denoted ΣX, is the quotient of $X \times I$ that identifies $X \times \{0\}$ to a point and $X \times \{1\}$ to a point (these are two different points!). Prove that if X is path-connected, then $\pi_1(\Sigma X) = 1$. Give an example to show that the path-connectedness is necessary.

3. Fix $n \geq 2$. Construct a connected space X whose fundamental group is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

4. Let
$$A = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 9\} \subset \mathbb{R}^2.$$
In other words, A is the annulus with inner radius 1 and outer radius 3. We will describe A using polar coordinates. Hence for $1 \leq r \leq 3$ and $\theta \in \mathbb{R}$, the point $(r, \theta) \in A$ corresponds to the point with x-coordinate $r\cos(\theta)$ and y-coordinate $r\sin(\theta)$. Define a map $f : A \to A$ by
$$f(r, \theta) = \begin{cases} (r, \theta + 2\pi(r-1)) & \text{if } 1 \leq r \leq 2, \\ (r, \theta - 2\pi(r-2)) & \text{if } 2 \leq r \leq 3. \end{cases}$$
This map is clearly continuous and $f(p) = p$ for p the point whose coordinates are $(x, y) = (2, 0)$.

   (a) Prove that f is homotopic to the identity map by a homotopy $F : A \times I \to A$ such that $F(q, t) = q$ if q lies on the "boundary" lines $r = 1$ and $r = 3$.

(b) Prove that the homotopy F from part a cannot fix the basepoint p. Hint : Embed A into a space X like the following:



The space X is the union of two annuli meeting along a square. Extend f by the identity to a map $g : X \to X$ such that $g|_A = f$ and investigate the effect of g on $\pi_1(X, p)$.