Madsen–Weiss for geometrically minded topologists

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We give an alternative proof of the Madsen–Weiss *generalized Mumford conjecture*; see Theorem 1.8. At the heart of the argument is a geometric version of *Harer stability*, which we formulate as a theorem about folded maps. A technical ingredient in the proof is an h–principle type statement, called the "wrinkling theorem" by the first and third authors [4]. Let us stress the point that we are neither proving the wrinkling theorem nor the Harer stability theorem.

Dedicated to D B Fuchs on the occasion of his 70th birthday

1 Introduction and statement of results

The Madsen–Weiss generalized Mumford conjecture says that a certain map

(1) $$\mathbb{Z} \times B\Gamma_\infty \to \Omega^\infty \mathbb{C}P^\infty_{-1}$$

is a homology equivalence, ie induces an isomorphism in integral homology. The map and the spaces $\mathbb{Z} \times B\Gamma_\infty$ and $\Omega^\infty \mathbb{C}P^\infty_{-1}$ will be defined below. The theorem was proved by Madsen and Weiss in [10] using homotopy theoretic techniques, such as simplicial spaces and homotopy colimits. In this paper we give an alternative proof of the Madsen–Weiss theorem, based on ideas similar to the original proof, but more geometrical and less homotopy theoretical in flavor. Instead of the homotopy theoretical techniques, we work directly with the underlying manifolds. After reformulating Harer's stability theorem [6] in a geometric form (see Theorem 1.9 below) as a statement about fibrations $f\colon M \to X$, the heart of our proof consists of generalizing this to a statement about certain *folded maps* (see Theorem 4.1) and applying a consequence of the wrinkling theorem (see Eliashberg and Mishachev [4]).

The geometric form of our main theorem gives a relation between *fibrations* (or *surface bundles*) and a related notion of *formal fibrations*. By a fibration we mean a smooth map $f\colon M \to X$, where M and X are smooth, oriented, compact manifolds and f is a

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submersion (ie $df\colon TM \to f^*TX$ is surjective). A cobordism between two fibrations $f_0\colon M_0 \to X_0$ and $f_1\colon M_1 \to X_1$ is a triple (W, Y, F) where W is a cobordism from M_0 to M_1, Y is a cobordism from X_0 to X_1, and $F\colon W \to Y$ is a submersion which extends $f_0 \amalg f_1$.

Definition 1.1 (i) An *unstable formal fibration* is a pair (f, φ), where $f\colon M \to X$ is a smooth proper map, and $\varphi\colon TM \to f^*TX$ is a bundle epimorphism.

(ii) A *stable formal fibration* (henceforth just a formal fibration) is a pair (f, φ), where f is as before, but φ is defined only as a *stable* bundle map. Thus for large enough j there is given an epimorphism $\varphi\colon TM \oplus \epsilon^j \to f^*TX \oplus \epsilon^j$, and we identify φ with its stabilization $\varphi \oplus \epsilon^1$. A cobordism between formal fibrations (f_0, φ_0) and (f_1, φ_1) is a quadruple (W, Y, F, Φ) which restricts to $(f_0, \varphi_0) \amalg (f_1, \varphi_1)$.

(iii) The formal fibration induced by a fibration $f\colon M \to X$ is the pair (f, df), and a formal fibration is *integrable* if it is of this form.

Our main theorem is about the case where $d = \dim M - \dim X = 2$. It relates the set of cobordism classes of fibrations with the set of cobordism classes of formal fibrations. Let us first discuss the *stabilization* process (or more precisely "stabilization with respect to genus". This should not be confused with the use of "stable" in "stable formal fibration". In the former, "stabilization" refers to increasing genus; in the latter it refers to increasing the dimension of vector bundles.) Suppose $f\colon M \to X$ is a formal fibration (we will often suppress the bundle epimorphism φ from the notation) and $j\colon X \times D^2 \to M$ is an embedding over X (ie $f \circ j = \mathrm{Id}\colon X \to X$), such that f is integrable on the image of j. Then we can *stabilize* f by taking the fiberwise connected sum of M with $X \times T$ (along j), where $T = S^1 \times S^1$ is the torus. If f happens to be integrable, this process increases the genus of each fiber by 1.

Our main theorem is the following.

Theorem 1.2 *Let $f\colon M \to X$ be a formal fibration with* $\dim M = 2 + \dim X$, *which is integrable over the image of a fiberwise embedding $j\colon X \times D^2 \to M$. Then, after possibly stabilizing a finite number of times, f is cobordant to an integrable fibration with connected fibers.*

We will also prove a relative version of the theorem. Namely, if X has boundary and f is already integrable, with connected fibers, over a neighborhood of ∂X, then the cobordism in the theorem can be also be assumed integrable over a neighborhood of the boundary.

Theorem 1.2 is equivalent to the Madsen–Weiss "generalized Mumford conjecture" [10] (ie the homology equivalence (1)). In the rest of this introduction we will explain why these are equivalent and introduce the methods that go into our proof of Theorem 1.2. Our proof is somewhat similar in ideas to the original proof, but quite different in details and language. The general scheme for reducing some algebro-topological problem to a problem of existence of bordisms between formal and genuine (integrable) fibrations was first proposed by D B Fuchs in [5]. That one might prove the Madsen–Weiss theorem in the form of Theorem 1.2 was also suggested by I Madsen and U Tillmann in [9].

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1.1 Diffeomorphism groups and mapping class groups

Let F be a compact oriented surface, possibly with boundary $\partial F = S$. Let $\operatorname{Diff}(F)$ denote the topological group of diffeomorphisms of F which restrict to the identity on the boundary. The classifying space $B\operatorname{Diff}(F)$ can be defined as the orbit space

$$B\operatorname{Diff}(F) = \operatorname{Emb}(F, \mathbb{R}^\infty)/\operatorname{Diff}(F),$$

and it is a classifying space for fibrations: for a manifold X, there is a natural bijection between isomorphism classes of smooth surface bundles $E \to X$ with fiber F and trivialized boundary $\partial E = X \times S$, and homotopy classes of maps $X \to B\operatorname{Diff}(F)$.

The *mapping class group* is defined as $\Gamma(F) = \pi_0 \operatorname{Diff}(F)$, ie the group of isotopy classes of diffeomorphisms of the surface. It is known that the identity component of $\operatorname{Diff}(F)$ is contractible (as long as $g \geq 2$ or $\partial F \neq \varnothing$), so $B\operatorname{Diff}(F)$ is also a classifying space for $\Gamma(F)$ (ie an Eilenberg–Mac Lane space $K(\Gamma(F), 1)$). When $\partial F = \varnothing$, this is also related to the *moduli space* of Riemann surfaces (ie the space of isomorphism classes of Riemann surfaces of genus g) via a map

$$B\operatorname{Diff}(F) \to \mathcal{M}_g \tag{2}$$

which induces an isomorphism in rational homology and cohomology. Mumford defined characteristic classes $\kappa_i \in H^{2i}(\mathcal{M}_g; \mathbb{Q})$ for $i \geq 1$ and conjectured that the resulting map

$$\mathbb{Q}[\kappa_1, \kappa_2, \dots] \to H^*(\mathcal{M}_g; \mathbb{Q})$$

is an isomorphism in degrees less than $(g-1)/2$. This is the original *Mumford Conjecture*.

It is convenient to take the limit $g \to \infty$. Geometrically, that can be interpreted as follows. Pick a surface $F_{g,1}$ of genus g and with one boundary component. Also pick an inclusion $F_{g,1} \to F_{g+1,1}$. Let T_∞ be the union of the $F_{g,1}$ over all g, ie a countably infinite connected sum of tori. We will consider fibrations $E \to X$ with trivialized "T_∞ ends".

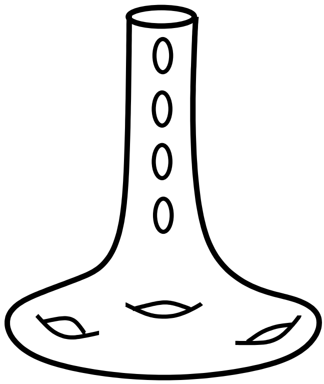


Figure 1: Surface with T_∞–end

Such a fibration is a pair (f, j) where $f\colon E \to X$ is a smooth fiber bundle with fiber T_∞, and

$$j\colon X \times T_\infty \rightsquigarrow E \tag{3}$$

is a *germ at infinity* of an embedding over X. This means, for X compact, that representatives of j are defined on the complement of some compact set in $X \times T_\infty$, and their images contain the complement of some compact set in E.

Let us describe a classifying space for fibrations with T_∞ ends. Let $B\Gamma_\infty$ be the mapping telescope (alias *homotopy colimit*) of the direct system

$$B\,\mathrm{Diff}(F_{0,1}) \to B\,\mathrm{Diff}(F_{1,1}) \to B\,\mathrm{Diff}(F_{2,1}) \to \cdots . \tag{4}$$

Lemma 1.3 *$\mathbb{Z} \times B\Gamma_\infty$ is a classifying space for fibrations with T_∞ ends.*

Proof Any compact $K \subset T_\infty$ is contained in $F_{g,1} \subset T_\infty$ for some finite g. Let $T_\infty^g \subset T_\infty$ be the complement of $F_{g,1}$. Let us consider for a moment fibrations with fiber T_∞ and an embedding as in (3), but which is actually defined on T_∞^k. Call such bundles *k–trivialized*. Specifying a k–trivialized bundle is the same thing as specifying a fibration $E' \to X$ with connected, compact fibers, and trivialized boundary $\partial E' = X \times S^1$ (namely E' is the complement of the image of j). Thus, the disjoint union

$$B = \coprod_g B\,\mathrm{Diff}(F_{g,1}) \tag{5}$$

is a classifying space for k–trivialized bundles (notice B is independent of k). A k–trivialized bundle is also a $(k+1)$–trivialized bundle, and increasing k is represented by a "stabilization" self-map $s\colon B \to B$. In the representation (5), s maps $B\,\mathrm{Diff}(F_{g,1})$ to $B\,\mathrm{Diff}(F_{g+1,1})$ and this is induced by the same map as in (4).

Now the statement follows by taking the direct limit as $k \to \infty$, and noticing that $\mathbb{Z} \times B\Gamma_\infty$ is the homotopy colimit of the direct system

$$B \xrightarrow{s} B \xrightarrow{s} \cdots . \qquad \square$$

1.2 A Thom spectrum

In this section we define a space $\Omega^\infty \mathbb{C}P^\infty_{-1}$ and interpret it as a classifying space for formal fibrations. The forgetful functor from fibrations to formal fibrations is represented by a map

$$B\,\mathrm{Diff}(F) \to \Omega^\infty \mathbb{C}P^\infty_{-1} . \tag{6}$$

We then consider the same situation, but where fibrations and formal fibrations have T_∞ ends. This changes the source of the map (6) to $\mathbb{Z} \times B\Gamma_\infty$, but turns out to not change the homotopy type of the target. We get a map

$$\mathbb{Z} \times B\Gamma_\infty \to \Omega^\infty \mathbb{C}P^\infty_{-1} ,$$

representing the forgetful functor from fibrations with T_∞ ends to formal fibrations with T_∞ ends.

The space $\Omega^\infty \mathbb{C}P^\infty_{-1}$ is defined as the Thom spectrum of the negative of the canonical complex line bundle over $\mathbb{C}P^\infty$. In more detail, let $\mathrm{Gr}_2^+(\mathbb{R}^N)$ be the Grassmannian manifold of oriented 2–planes in $\mathbb{R}^N$. It supports a canonical 2–dimensional vector bundle γ_N with an $(N-2)$–dimensional complement $\gamma_N^\perp$ such that $\gamma_N \oplus \gamma_N^\perp = \epsilon^N$. There is a canonical identification

$$\gamma_{N+1}^\perp | \mathrm{Gr}_2^+(\mathbb{R}^N) = \gamma_N^\perp \oplus \epsilon^1. \tag{7}$$

The Thom space $\mathrm{Th}(\gamma_N^\perp)$ is defined as the one-point compactification of the total space of $\gamma_N^\perp$, and the identification (7) induces a map $S^1 \wedge \mathrm{Th}(\gamma_N^\perp) \to \mathrm{Th}(\gamma_{N+1}^\perp)$. The space $\Omega^\infty \mathbb{C}P^\infty_{-1}$ is defined as the direct limit

$$\Omega^\infty \mathbb{C}P^\infty_{-1} = \lim_{N\to\infty} \Omega^N \mathrm{Th}(\gamma_N^\perp).$$

Like we did for $B\,\mathrm{Diff}(F)$, we shall think of $\Omega^\infty \mathbb{C}P^\infty_{-1}$ as a classifying space, ie interpret homotopy classes of maps $X \to \Omega^\infty \mathbb{C}P^\infty_{-1}$ from a smooth manifold X in terms of certain geometric objects over X. Recall the notion of *formal fibration* from

Definition 1.1 above. A cobordism (W, Y, F, Φ) of formal fibrations is a *concordance* if the target cobordism is a cylinder: $Y = X \times [0,1]$.

Lemma 1.4 *There is a natural bijection between set*

$$[X, \Omega^\infty \mathbb{C}P^\infty_{-1}]$$

of homotopy classes of maps, and the set of concordance classes of formal fibrations over X.

Proof sketch This is the standard argument of Pontryagin–Thom theory. In one direction, given a map $X \to \Omega^N \mathrm{Th}(\gamma_N^\perp)$, one makes the adjoint map $g\colon X \times \mathbb{R}^N \to \mathrm{Th}(\gamma_N^\perp)$ transverse to the zero section of $\gamma_N^\perp$ and sets $M = g^{-1}(\text{zero-section})$. Then M comes with a map $c\colon M \to \mathrm{Gr}_2^+(\mathbb{R}^N)$ and the normal bundle of $M \subset X \times \mathbb{R}^N$ is $c^*(\gamma_N^\perp)$. This gives a stable isomorphism $TM \cong_{\mathrm{st}} f^*TX \oplus c^*(\gamma_N)$ and hence a stable epimorphism $TM \to TX$.

In the other direction, given a formal fibration (f, φ) with $f\colon M \to X$, we pick an embedding $M \subset X \times \mathbb{R}^N$. Letting ν be the normal bundle of this embedding, we get a "collapse" map

$$X_+ \wedge S^N \to \mathrm{Th}(\nu). \tag{8}$$

We also get an isomorphism of vector bundles over M

$$TM \oplus \nu \cong f^*TX \oplus \epsilon^N. \tag{9}$$

Let $\xi\colon M \to \mathrm{Gr}_2^+(\mathbb{R}^N)$ be a classifying map for the kernel of the stable epimorphism $\varphi\colon TM \to f^*TX$ (this is a two-dimensional vector bundle with orientation induced by the orientations of X and M), so we have a stable isomorphism $TM \cong_{\mathrm{st}} f^*TX \oplus \xi^*\gamma_N$. Combining with (9) we get a stable isomorphism $\xi^*\gamma_N \oplus \nu \cong_{\mathrm{st}} \epsilon^N$. By adding $\xi^*(\gamma_N^\perp)$ we get a stable isomorphism $\nu \cong_{\mathrm{st}} \xi^*(\gamma_N^\perp) \oplus \epsilon^N$ which we can assume is induced by an unstable isomorphism (since we can assume $N \gg \dim M$)

$$\nu \cong \xi^*\gamma_N^\perp. \tag{10}$$

This gives a proper map $\nu \to \gamma_N^\perp$ and hence a map of Thom spaces $\mathrm{Th}(\nu) \to \mathrm{Th}(\gamma_N^\perp)$. Compose with (8) and take the adjoint to get a map $X \to \Omega^N \mathrm{Th}(\gamma_N^\perp)$. Finally let $N \to \infty$ to get a map $X \to \Omega^\infty \mathbb{C}P^\infty_{-1}$.

The homotopy class of the resulting map $X \to \Omega^\infty \mathbb{C}P^\infty_{-1}$ is well defined and depends only on the concordance class of the formal fibration $f\colon M \to X$. □

A fibration naturally gives rise to a formal fibration, and this association gives rise to a map of classifying spaces which is the map (2). We would like to make this process compatible with the stabilization procedure explained in Section 1.1 above. To this end we consider formal fibrations with k–trivialized T_∞ ends. This means that $f\colon M \to X$ is equipped with an embedding over X

$$j\colon X \times T^k_\infty \to M,$$

and that (f, φ) is integrable on the image of j. Of course, we also replace the requirement that M be compact by the requirement that the complement of the image of j be compact.

Lemma 1.5 *Formal fibrations with k–trivialized ends are represented by the space $\Omega^\infty \mathbb{C}P^\infty_{-1}$.*

Proof sketch This is similar to the proof of Lemma 1.4 above. Applying the Pontryagin–Thom construction from the proof of that lemma to the projection $X \times T^k_\infty \to X$ gives a path $\alpha_0\colon [k, \infty) \to \Omega^{N-1}\mathrm{Th}(\gamma_N^\perp)$. Applying the Pontryagin–Thom construction to an arbitrary k–trivialized formal fibration gives a path $\alpha\colon [0, \infty) \to \Omega^{N-1}\mathrm{Th}(\gamma_N^\perp)$ whose restriction to $[k, \infty)$ is α_0. The space of all such paths is homotopy equivalent to the loop space $\Omega^N \mathrm{Th}(\gamma_N^\perp)$. □

Increasing k gives a diagram of stabilization maps

$$\begin{array}{ccc} \coprod_g B\,\mathrm{Diff}(F_{g,1}) & \longrightarrow & \Omega^\infty \mathbb{C}P^\infty_{-1} \\ \big\downarrow{\scriptstyle s} & & \big\downarrow{\scriptstyle s} \\ \coprod_g B\,\mathrm{Diff}(F_{g,1}) & \longrightarrow & \Omega^\infty \mathbb{C}P^\infty_{-1}. \end{array}$$

On the right hand "formal" side, the stabilization map is up to homotopy described as multiplication by a fixed element (multiplication in the loop space structure. See the proof of Lemma 1.5.) This is a homotopy equivalence, with inverse given by multiplication by a point in the path component of $\Omega^\infty \mathbb{C}P^\infty_{-1}$ which is inverse to $[s] \in \pi_0(\Omega^\infty \mathbb{C}P^\infty_{-1})$. Therefore the direct limit has the same homotopy type $\Omega^\infty \mathbb{C}P^\infty_{-1}$. Taking the direct limit we get the desired map

$$\mathbb{Z} \times B\Gamma_\infty \to \Omega^\infty \mathbb{C}P^\infty_{-1}.$$

This is the map (1). The target of this map should be thought of as the homotopy direct limit of the system

$$\Omega^\infty \mathbb{C}P^\infty_{-1} \xrightarrow{s} \Omega^\infty \mathbb{C}P^\infty_{-1} \xrightarrow{s} \cdots$$

and as a classifying space for formal fibrations with T_∞ ends. (It is a "coincidence" that the classifying space for formal fibrations and the classifying space for formal fibrations with T_∞ ends have the same homotopy type).

1.3 Oriented bordism

For a pair (A, B) of spaces, oriented bordism $\Omega_n(A,B) = \Omega_n^{\mathrm{SO}}(A,B)$ is defined as the set of bordism classes of continuous maps of pairs

$$f\colon (X, \partial X) \to (A, B)$$

for smooth oriented compact n–manifolds X with boundary ∂X. To be precise, a bordism between two maps $f_\pm\colon (X_\pm, \partial X_\pm) \to (A,B)$ is a map $F\colon (W, \partial' W) \to (A, B)$, where W is a compact, oriented manifold with boundary with corners, so that $\partial W = \partial_- W \cup \partial' W \cup \partial_+ W$, where $\partial_\pm W = X_\pm$ and $\partial' W$ is a cobordism between closed manifolds ∂X_- and ∂X_+, and the map $F\colon (W, \partial' W) \to (A, B)$ such that $F|_{\partial_\pm W} = f_\pm$. In particular, a map $f\colon (X, \partial X) \to (A, B)$ represents the zero element of $\Omega_n(A, B)$ if and only if there exists $F\colon W^{n+1} \to A$ with $X \subset \partial W$ and $F(\partial W - \operatorname{Int} X) \subset B$.

For a single space A set $\Omega_n(A) = \Omega_n(A, \varnothing)$. Oriented bordism is a *generalized homology theory*. This means that it satisfies the usual Eilenberg–Steenrod axioms for homology (long exact sequence etc) except for the dimension axiom. In particular a map $B \to A$ induces an isomorphism $\Omega_*(B) \to \Omega_*(A)$ if and only if the relative groups $\Omega_*(A, B)$ all vanish. The following result is well known to follow from the Atiyah–Hirzebruch spectral sequence (for completeness we give a geometric proof in Appendix B).

Lemma 1.6 *Let $f\colon B \to A$ be a continuous map of topological spaces. Then the following statements are equivalent.*

(i) *$f_*\colon H_k(B) \to H_k(A)$ is an isomorphism for $k < n$ and an epimorphism for $k = n$.*

(ii) *$f_*\colon \Omega_k(B) \to \Omega_k(A)$ is an isomorphism for $k < n$ and an epimorphism for $k = n$.*

In particular, f induces an isomorphism in homology in all degrees if and only if it does so in oriented bordism.

We apply this lemma to the map $\mathbb{Z} \times B\Gamma_\infty \to \Omega^\infty \mathbb{C}P^\infty_{-1}$. Interpreting $\mathbb{Z} \times B\Gamma_\infty$ and $\Omega^\infty \mathbb{C}P^\infty_{-1}$ as classifying spaces for fibrations, resp. formal fibrations, with T_∞ ends, we get the following interpretation (using that $\Omega_n(A, B)$ can also be defined in terms of maps $f\colon X \to A$ such that $f^{-1}(B)$ is a *neighborhood* of $\partial X \subset X$).

Lemma 1.7 *There is a natural bijection between the relative oriented bordism groups* $\Omega_*(\Omega^\infty \mathbb{C}P^\infty_{-1}, \mathbb{Z} \times B\Gamma_\infty)$ *and cobordism classes of formal fibrations* $f\colon M \to X$ *with* T_∞ *ends. The formal fibration is required to be integrable over a neighborhood of* ∂X*, and cobordisms* $F\colon E \to W$ *are required to be integrable over a neighborhood of* $\partial' W$*.*

That (1) induces an isomorphism in integral homology (the Madsen–Weiss theorem) is now, by Lemma 1.6, equivalent to the statement that the relative groups

$$\Omega_*(\Omega^\infty \mathbb{C}P^\infty_{-1}, \mathbb{Z} \times B\Gamma_\infty)$$

all vanish. By Lemma 1.7 this is equivalent to:

Theorem 1.8 *Any formal fibration* $f\colon M \to X$ *with* T_∞ *ends is cobordant to an integrable one. More precisely, there exists a formal fibration* $F\colon E \to W$ *with* T_∞ *ends, which restricts to* f *over* $X \subset \partial W$*, and which is integrable over* $\partial W - \mathrm{Int}(X)$*.*

Theorem 1.8 is our main result. It is a geometric version of the Madsen–Weiss theorem. It is obviously equivalent to Theorem 1.2 above (with its relative form).

1.4 Harer stability

J Harer proved a homological stability theorem in [6] which implies precise bounds on the number of stabilizations needed in Theorem 1.2. At the same time, it will play an important role in the proof of that theorem (as it does in [10]).

Roughly it says that the homology of the mapping class group of a surface F is independent of the topological type of F, as long as the genus is high enough. The result was later improved by Ivanov [7; 8] and then by Boldsen [2]. We state the precise result.

Consider an inclusion $F \to F'$ of compact, connected, oriented surfaces. Let $S = \partial F'$, and let $\Sigma \subset F'$ denote the complement of F. Thus $F' = F \cup_{\partial F} \Sigma$. There is an induced map of classifying spaces

$$B\,\mathrm{Diff}(F) \to B\,\mathrm{Diff}(F'). \tag{11}$$

A map $f\colon X \to B\,\mathrm{Diff}(F')$ classifies a fibration $E \to X$ with fiber F' and boundary $\partial E = X \times S$, where $S = \partial F'$. Lifting it to a map into $B\,\mathrm{Diff}(F)$ amounts to extending the embedding $X \times S \to E$ to an embedding

$$X \times \Sigma \to E$$

over X.

The most general form of Harer stability states that the map (11) induces an isomorphism in $H_k(-;\mathbb{Z})$ for $k < 2(g-1)/3$, where g is the genus of F. Consequently, by Lemma 1.6, it induces an isomorphism in oriented bordism $\Omega_n(-)$ for $n < 2(g-1)/3$ or, equivalently, the relative bordism group

$$\Omega_n(B\,\mathrm{Diff}(F'), B\,\mathrm{Diff}(F))$$

vanishes for $n < 2(g-1)/3$. Thus, Harer stability has the following very geometric interpretation: For any fibration $f\colon E \to X$ with fiber F' and boundary $\partial E = X \times S$, f is cobordant to a fibration $f'\colon E' \to X'$ via a cobordism $F\colon W \to M$ (a fibration with trivialized boundary $M \times S$) where the trivialization $X' \times S = \partial E' \to E'$ extends to an embedding

$$X' \times \Sigma \to E'. \tag{12}$$

Moreover, the trivialization (12) can be assumed compatible with any given extension $(\partial X) \times \Sigma \to E$ over $\partial X = \partial X'$. Here we assume $F' = F \cup_{\partial F} \Sigma$ as above, that F and F' are connected, and that F has large genus. If the fibration has T_∞ ends, the genus assumption is automatically satisfied, and we get the following geometric consequence of Harer's theorem. (We remind the reader that we have merely rephrased Harer's theorem [6], we have not given an independent proof.)

Theorem 1.9 (Geometric form of Harer stability) *Let $\Sigma_1 \subset \Sigma_2$ be compact surfaces with boundary (the surfaces are not assumed connected). Let $f\colon M \to X$ be a fibration with T_∞ ends, and let*

$$j\colon (\partial X \times \Sigma_2) \cup (X \times \Sigma_1) \to M$$

be a fiberwise embedding over X, such that in each fiber the complement of its image is connected. Then, after possibly changing $f\colon M \to X$ by a bordism which is the trivial bordism over ∂X, the embedding j extends to an embedding of $X \times \Sigma_2$.

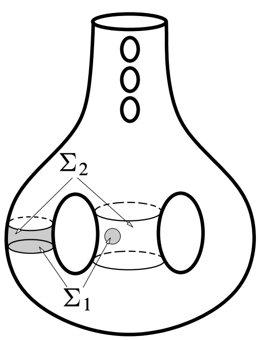


Figure 2: $\Sigma_1 \subset \Sigma_2 \subset F$

Explicitly, the bordism in the theorem is a fibration $F\colon W \to Y$ with T_∞ ends, where the boundary ∂Y is partitioned as $\partial Y = X \cup X'$ with $\partial X = \partial X' = X \cap X'$ and $F_{|X} = f$. The extension of j is a fiberwise embedding

$$J\colon (Y \times \Sigma_1) \cup (X' \times \Sigma_2) \to W$$

over Y.

1.5 Outline of proof of Theorem 1.8

1.5.1 From formal fibrations to folded maps The overall aim given a formal fibration $(f\colon M \to X, \varphi)$ with T_∞ ends, is to get rid of all singularities of f after changing it via *bordisms*. Our first task will be to simplify the singularities of f as much as possible using only *homotopies*. The simplest generic singularities of a map $f\colon M \to X$ are *folds*. A map with only fold singularities is called a *folded map*. The fold locus $\Sigma(f)$ consists of points where the rank of f is equal to $\dim X - 1$, while the restriction $f|_{\Sigma(f)}\colon \Sigma(f) \to M$ is an immersion. In the case when $\dim M = \dim X + 2 = n+2$, which is the case we consider in this paper, we have $\dim \Sigma(f) = n-1$. A certain additional structure on folded maps, called an *enrichment*, allows one to define a homotopically canonical *suspension*, ie a bundle epimorphism $\varphi = \varphi_f\colon TM \oplus \epsilon^1 \to TX \oplus \epsilon^1$, such that (f, φ) is a formal fibration. The enrichment of a folded map f consists of

- an n–dimensional submanifold $V \subset M$ such that $\partial V = \Sigma(f)$, and the restriction of f to each connected component of $V_i \subset V$ is an embedding $\operatorname{Int} V_i \to X$;
- a trivialization of the bundle $\operatorname{Ker} df|_{\operatorname{Int} V}$ with a certain additional condition on the behavior of this trivialization on $\partial V = \Sigma(f)$.

Of course, existence of an enrichment is a strong additional condition on the fold map. In Section 2.3, we explain how to associate to an enriched folded map (f, ε) a formal fibration $(f, \mathcal{L}(f, \varepsilon))$, where $\mathcal{L}(f, \varepsilon)\colon TM \oplus \epsilon^1 \to TX \oplus \epsilon^1$ is a bundle epimorphism associated to the enrichment ϵ. The main result of Section 2 is Theorem 2.17, which proves that any formal fibration can be represented in this way (plus a corresponding relative statement). This is proved using an h–principle type result, the "wrinkling theorem", proven by the first and third authors [4]. Note that Theorem 2.17 is a variation of the main result of the first author [3] and can also be proven by the methods of that paper. Also in Section 2 we recall some basic facts about folds and other simple singularities of smooth maps, and discuss certain surgery constructions needed for the rest of the proof of Theorem 1.8. This part works independently of the codimension $d = \dim N - \dim M$, and hence the exposition in this section is done for arbitrary $d > 0$.

1.5.2 Getting rid of elliptic folds For a folded map $f\colon M \to X$ with folds $\Sigma(f) \subset M$, each path component of $\Sigma(f)$ has an index which is well defined provided that the immersion $\Sigma(f) \to X$ is cooriented (ie there is a chosen trivialization of the normal bundle). Assuming this is done, folds in the case $d = 2$ can be of index $0, 1, 2$ and 3. We call folds of index $1, 2$ *hyperbolic* and folds of index $0, 3$ *elliptic*. It is generally impossible to get rid of elliptic folds by a homotopy of the map f. However, it is easy to do so if one allows to change f to a *bordant* map $\widetilde{f}\colon \widetilde{M} \to X$. This bordism trades each elliptic fold component by a parallel copy of a hyperbolic fold; see Figure 15 and Section 3.3 below. A similar argument allows one to make all fibers $\widetilde{f}^{-1}(x), x \in X$, connected.

1.5.3 Generalized Harer stability theorem A generalization of Harer's stability theorem, in the geometric form Theorem 1.9, plays an important role in our proof. The generalization consists of allowing the map $f\colon M \to X$ to have fold singularities. We still require that f has T_∞ ends, that the nonsingular fibers $f^{-1}(x) - \Sigma(f)$ are connected, and that the folds are enriched (as in Section 1.5.1 above). The generalized Harer stability theorem (Theorem 4.1 below) is similar to Theorem 1.9, but starts with an enriched folded map f.

In Section 4.3 below we deduce Theorem 4.1 from Harer's Theorem 1.9. The idea of the proof is not hard to explain. Given a folded map $f\colon M \to X$, the restriction of f to the fold $\Sigma(f) \subset M$ is a codimension 1 immersion $\Sigma(f) \to X$. After a small perturbation we can assume that $\Sigma(f) \to X$ is self-transverse and has normal crossings, and we get a stratification of X by multiplicity of self-intersection. The map $f\colon M \to X$ restricts to a fiber bundle over each open stratum, the fiber is a surface with as many singularities as the codimension of the stratum. The proof of the generalized Harer stability theorem proceeds by induction over the strata. Over each stratum we apply Theorem 1.9 to produce a bordism of the stratum, which by a gluing procedure gives a cobordism of $f\colon M \to X$.

The generalized Harer stability theorem is used in a crucial way in the next step, getting rid of hyperbolic folds. This is explained in outline in a very special case in the next section.

1.5.4 Getting rid of hyperbolic folds Let f be an enriched folded map with hyperbolic folds and with connected fibers. Let C be one of the fold components and $\overline{C} = f(C) \subset X$ its image (the enrichment ensures that $\overline{C}$ has no self-intersections). For the purpose of this introduction we will consider only the following special case. First, we will assume that C is homologically trivial. As we will see, when $\dim X > 1$ this will always be possible to arrange. In particular, $\overline{C}$ bounds a domain $U_C \subset X$. Next, we will assume that the fold C has index 1 with respect to the outward coorientation of the boundary of the domain U_C. In other words, when the point $x \in X$ travels across $\bar{C}$ into U_C then one of the circles in the fiber $f^{-1}(x)$ collapses to a point, so locally the fiber gets disconnected to two disks; see Figure 3. The inverse index 1 surgery makes a connected sum of two disks at their centers. Note that on an open collar

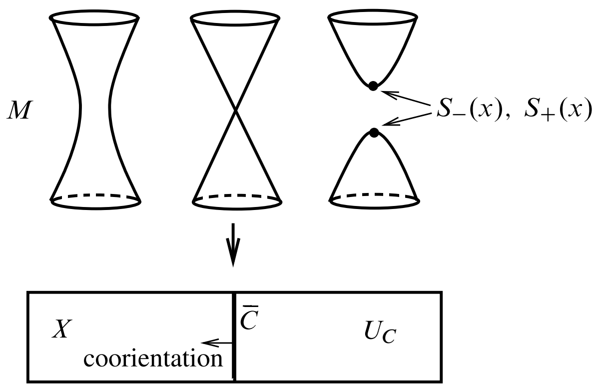


Figure 3: Fibers of an index 1 fold

$\Omega = \partial U_C \times (0,1) \subset \operatorname{Int} U_C$ along C in U_C there exists two sections $S_\pm\colon \Omega \to M$ such that the 0–sphere $\{S_-(x), S_+(x)\}$ is the "vanishing cycle", for the index 1 surgery when x travels across $\bar{C}$. Moreover, the enrichment structure ensures that the vertical bundle along these sections is trivial. If one could extend the sections $S_\pm$ to all of U_C preserving all these properties, then the fiberwise index 1 surgery, attaching 1–handle along small disks surrounding $S_\pm(x)$ and $S_\pm(x)$, $x \in U_C$, would eliminate the fold $\bar{C}$. This is one of the fold surgeries described in detail in Section 3.2.

Though such extensions $S_\pm(x)\colon \operatorname{Int} U_C \to M$ need not exist for our original folded map f, Harer's stability theorem in the form of Theorem 1.9 states that there is an enriched folded map $\widetilde{f}\colon \widetilde{M} \to \widetilde{X}$, bordant to f, for which such sections do exist, and hence the fold $\bar{C}$ could be eliminated.

1.6 Organization of the paper

As already mentioned, Section 2 recalls basic definitions and necessary results and constructions involving fold singularities. In Section 2.1 we define folded maps. The goal of Section 2.2 is Theorem 2.4 which is an h–principle for constructing so-called *special folded* maps, whose folds are organized in pairs of spheres. This theorem is a reformulation of the wrinkling theorem from [4]. We deduce Theorem 2.4 from the wrinkling theorem in Appendix A. In Section 2.3 we define the notion of enrichment for folded maps and prove that an enriched folded map admits a homotopically canonical suspension and hence gives rise to a formal fibration. The rest of Section 2 will prove that any formal fibration is cobordant to one induced by an enriched folded map.

The input of Theorem 2.4 is a map $f\colon M \to X$ and a fiberwise surjective map $\varphi\colon TM \to TX$ between the *nonstabilized* tangent bundles. Formal fibrations are defined in terms of *stable* epimorphisms, and in Section 2.4 we explain the modifications necessary in the stable case. In Section 2.5 we formulate and prove Theorem 2.17 which reduces formal fibrations to enriched folded maps.

In Section 3.1 we introduce several special bordism categories and formulate the two remaining steps of the proof: Proposition 3.6 which allows us to get rid of elliptic folds, and Proposition 3.7 which eliminates the remaining hyperbolic folds. Section 3.2 is devoted to fold surgery constructions which are needed in the proof of Propositions 3.6 and 3.7. These are just fiberwise Morse surgeries, in the spirit of surgery of singularities techniques developed in [3]. Proposition 3.6 is proved in Section 3.3.

Section 4 begins with the proof of our generalized Harer stability theorem for folded maps (Theorem 4.1) in Section 4.1 and Section 4.2. We conclude the proof of our main result, Theorem 1.8, by proving Proposition 3.7 about eliminating of hyperbolic folds in Section 4.4. In Section 5 we collect two Appendices. In Appendix A we deduce Theorem 2.4 from the wrinkling theorem from [4]. Appendix B is devoted to a geometric proof of Lemma 1.6.

2 Folded maps

2.1 Folds

Let M and X be smooth manifolds of dimension $m = n + d$ and n, respectively. Although the applications in this paper require only the case $d = 2$, the discussion is Section 2 (except for Section 3.2.4) applies equally well for any $d \geq 0$. For a smooth map $f\colon M \to X$ we will denote by $\Sigma(f)$ the set of singular points, ie

$$\Sigma(f) = \left\{p \in M,\ \operatorname{rank} d_p f < n\right\}.$$

A point $p \in \Sigma(f)$ is called a *fold* of index k if near the point p the map f is equivalent to the map

$$\mathbb{R}^{n-1} \times \mathbb{R}^{d+1} \to \mathbb{R}^{n-1} \times \mathbb{R}^1$$

given by the formula

$$(y, x) \mapsto \left(y,\ Q(x) = -\sum_1^k x_i^2 + \sum_{k+1}^{d+1} x_j^2\right) \tag{13}$$

where $x = (x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1}$ and $y = (y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1}$. We will also denote $x_- = (x_1, \dots, x_k)$, $x_+ = (k_{k+1}, \dots, x_{d+1})$ and write $Q(x) = -|x_-|^2 + |x_+|^2$. For $M = \mathbb{R}^1$ this is just a nondegenerate index k critical point of the function $f\colon V \to \mathbb{R}^1$. By a *folded map* we will mean a map with only fold singularities. Given $y \in \mathbb{R}^{n-1}$ and an $\epsilon > 0$, the $(k-1)$–dimensional sphere $y \times \{Q(x) = -\epsilon, x_+ = 0\} \subset \mathbb{R}^{n-1} \times \mathbb{R}^{d+1}$ is the *vanishing cycle* of the fold over the point $(y, -\epsilon)$.

Let $C \subset \Sigma(f)$ be a path component of the folds of $f\colon M \to X$. The restriction $f\colon C \to X$ is an immersion and the normal bundle is a real line bundle over C. A *coorientation* of $\bar{C} = f(C)$ is a trivialization of this line bundle. A choice of coorientation allows one to provide each fold component C with a well-defined index s, which changes from s to $d+1-s$ with a switch of the coorientation. The normal bundle of $C \subset M$ is $\operatorname{Ker} df$, and the second derivatives of f gives an invariantly defined nondegenerate quadratic form $d^2 f\colon \operatorname{Ker} df|_C \to \operatorname{Coker} df$. The coorientation of C yields a trivialization of $\operatorname{Coker}(df)$ and thus $d^2 f$ can be viewed as a real-valued quadratic form on $\operatorname{Ker} df$. We shall write $\operatorname{Cone}_\pm(C) \subset \operatorname{Ker}(df|_C)$ for the open sets $\{z \in \operatorname{Ker} df; \pm d^2 f(z) > 0\}$. There is a splitting of vector bundles

$$\operatorname{Ker} df|_C = \operatorname{Ker}_-(C) \oplus \operatorname{Ker}_+(C),$$

defined uniquely up to homotopy by the condition $\operatorname{Ker}_\pm(C) \setminus 0 \subset \operatorname{Cone}_\pm(C)$.

2.2 Double folds and special folded mappings

Given an orientable $(n-1)$–dimensional manifold C, let us consider the map

$$w_C(n+d, n, k)\colon C \times \mathbb{R}^1 \times \mathbb{R}^d \to C \times \mathbb{R}^1 \tag{14}$$

given by the formula

$$(y, z, x) \mapsto \left(y, z^3 - 3z - \sum_1^k x_i^2 + \sum_{k+1}^d x_j^2 \right), \tag{15}$$

where $y \in C$, $z \in \mathbb{R}^1$ and $x \in \mathbb{R}^d$. This is a folded map, with folds given by

$$\Sigma(w_C) = C \times S^0 \times \{0\} \subset C \times \mathbb{R}^1 \times \mathbb{R}^d,$$

two copies of the manifold C. The manifold $C \times \{1\}$ is a fold of index k, while $C \times \{-1\}$ is a fold of index $k+1$, with respect to the coorientation of the folds in the image given by the vector field $\partial/\partial u$ where u is the coordinate $C \times \mathbb{R} \to \mathbb{R}$. It is important to notice that the restriction of the map $w_C(n+d, n, k)$ to the annulus

$$A = C \times \operatorname{Int} D^1 = C \times \operatorname{Int} D^1 \times 0 \subset C \times \mathbb{R}^1 \times \mathbb{R}^d$$

is an *embedding*. Figure 4 illustrates the case $C = S^1$.

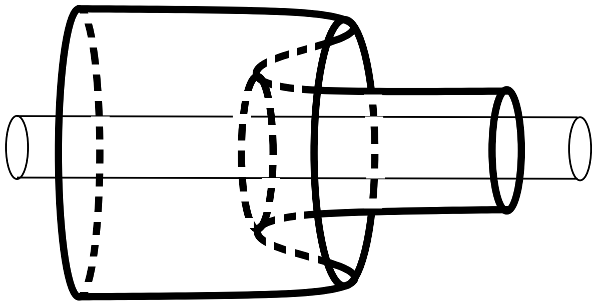


Figure 4: The radial projection to the cylinder has a double fold along $C = S^1$

Although the differential

$$dw_C(n+d,n,k)\colon T(C\times\mathbb{R}^1\times\mathbb{R}^d)\to T(C\times\mathbb{R}^1)$$

degenerates over $\Sigma(w_C)$, it can be canonically *regularized* over $\mathcal{O}p(C\times D^1)$, an open neighborhood of the annulus $C\times D^1$. Namely, we can change the element $3(z^2-1)$ in the Jacobi matrix of $w_C(n+d,n,k)$ to a positive function γ, which coincides with $3(z^2-1)$ on $\mathbb{R}^1\setminus[-1-\delta,1+\delta]$ for sufficiently small δ. The new bundle map

$$\mathcal{R}(dw_C)\colon T(C\times\mathbb{R}^1\times\mathbb{R}^d)\to T(C\times\mathbb{R}^1)$$

provides a homotopically canonical extension of the map

$$dw_C\colon T(C\times\mathbb{R}^1\times\mathbb{R}^d\setminus\mathcal{O}p(C\times D^1))\to T(C\times\mathbb{R}^1)$$

to an epimorphism (fiberwise surjective bundle map)

$$T(C\times\mathbb{R}^1\times\mathbb{R}^d)\to T(C\times\mathbb{R}^1).$$

We call $\mathcal{R}(dw_C)$ the *regularized differential* of the map $w_C(n+d,n,k)$.

A map $f\colon U\to X$ defined on an open set $U\subset M$ is called a *double C–fold* of index $k+1/2$ if it is equivalent to the restriction of the map $w_C(n+d,n,k)$ to an open neighborhood of $C\times D^1$. For instance, when $X=\mathbb{R}$ and C is a point, a double C–fold is a Morse function given in a neighborhood of a gradient trajectory connecting two critical points of neighboring indices. In the case of general n, a double C–fold is called a *spherical double fold* if $C=S^{n-1}$.

It is always easy to create a double C–fold as the following lemma shows. The lemma is a parametric version of the creation of a canceling pair of critical points of a Morse function.

Lemma 2.1 *Given a submersion $f\colon U \to X$ of a manifold U, a closed submanifold $C \subset U$ of dimension $n-1$ such that $f|_C\colon C \to X$ is an embedding with trivialized normal bundle, and a splitting $K_- \oplus K_+$ of the vertical bundle* $\mathrm{Vert} = \mathrm{Ker}\, df$ *into two trivialized subbundles $K_+, K_- \subset K$ over $\mathcal{O}p\, C$, one can construct a map $\widetilde{f}\colon U \to X$ such that*

- *$\widetilde{f}$ coincides with f near ∂U;*
- *in a neighborhood of C the map $\widetilde{f}$ has a double C–fold, ie it is equivalent to the map (15) restricted to an open neighborhood of $A = C \times D^1$, such that the frames*
$$\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right) \quad \text{and} \quad \left(\frac{\partial}{\partial x_{k+1}}, \dots, \frac{\partial}{\partial x_d}\right)$$
along A provide the given trivializations of the bundles K_- and K_+;
- *df and $\mathcal{R}(d\widetilde{f})$ are homotopic via a homotopy of epimorphisms fixed near ∂U.*

Proof There exist neighborhoods $U_1 \supseteq C$ and $U_2 \supseteq \overline{C} = f(C)$ and parametrizations $U_1 = C \times [-2,2] \times D^d$, $U_2 = \overline{C} \times [-2,2]$, such that f has the form

$$C \times [-2,2] \times [-1,1]^d \ni (v, z, x = (x_1, \dots, x_d)) \mapsto (\overline{v} = f(v), z) \in \overline{C} \times [-2,2],$$

and the frames $(\partial/\partial x_1, \dots, \partial/\partial x_k)$ and $(\partial/\partial x_{k+1}, \dots, \partial/\partial x_d)$ along A provide the given trivializations of the bundles K_- and K_+. Consider a C^∞ function $\lambda\colon [-2,2] \to [-2,2]$ which coincides with $z^3 - 3z$ on $[-1,1]$, with z near ± 2, and such that ± 1 are its only critical points. Take two cut-off C^∞–functions $\alpha, \beta\colon \mathbb{R}^+ \to [0,1]$ such that $\alpha = 1$ on $[0, 1/4]$, $\alpha = 0$ on $[1/2, 1]$, $\beta = 1$ on $[0, 1/2]$ and $\beta = 0$ on $[3/4, 1]$. Set $Q(x) = -\sum_1^k x_i^2 + \sum_{k+1}^d x_j^2$ and define first a map $\widehat{f}\colon U_1 \to U_2$ by the formula

$$\widehat{f}(v, z, x) = (f(v), \alpha(|x|)\lambda(z) + (1 - \alpha(|x|))z + \beta(|x|)Q(x)),$$

and then extend it to M, being equal to f outside $U_1 \subset U$. The regularized differential of a linear deformation between f an $\widehat{f}$ provides the required homotopy between df and $\mathcal{R}(d\widetilde{f})$. □

Remark 2.2 It could be difficult to eliminate a double C–fold. Even in the case $n = 1$ this is one of the central problems of Morse theory, leading to eg the h–cobordism theorem.

Definition 2.3 A map $f\colon M \to X$ is called *special folded*, if there exist disjoint open subsets $U_1, \dots, U_l \subset M$ such that the restriction $f|_{M \setminus U}$, $U = \bigcup_1^l U_i$, is a submersion (ie has rank equal n) and for each $i = 1, \dots, l$ the restriction $f|_{U_i}$ is a spherical double fold. *In addition,* we require that the images of all fold components bound balls in X.

The singular locus $\Sigma(f)$ of a special folded map f is a union of $(n-1)$–dimensional double spheres $S^{n-1}\times S^0_{(i)}=\Sigma(f|_{U_i})\subset U_i$. By definition, each double sphere $S^{n-1}\times S^0_i$ is the boundary of an annulus $A_i=S^{n-1}\times D^1\subset U_i$. Notice that although the restriction of f to each annulus $S^{n-1}\times \operatorname{Int} D^1_{(i)}$ is an *embedding*, the restriction of f to the union of all the annuli $S^{n-1}\times \operatorname{Int} D^1_{(i)}$ is, in general, only an *immersion*, because the images of the annuli may intersect each other. Using an appropriate version of the transversality theorem we can arrange by a C^∞–small perturbation of f that all combinations of images of its fold components intersect transversally. The differential $df\colon TM\to TX$ can be regularized to obtain an epimorphism $\mathcal{R}(df)\colon TM\to TX$. To get $\mathcal{R}(df)$ we regularize $df|_{U_i}$ for each double fold $f|_{U_i}$.

In our proof of Theorem 1.8 we will use the following result about special folded maps, which is a modification of the wrinkling theorem from [4]. Its proof is given in Appendix A, where we derive it from the wrinkling theorem (see Theorem 5.1).

Theorem 2.4 (Special folded mappings) *Let $F\colon TM\to TX$ be a fiberwise epimorphism which covers a map $f\colon M\to X$. Suppose that f is a submersion on a neighborhood of a closed subset $K\subset M$, and that F coincides with df over that neighborhood. Then if* $\dim M>\dim X$ *there exists a special folded map $g\colon M\to X$ which coincides with f near K and such that $\mathcal{R}(dg)$ and F are homotopic rel $TM|_K$ through fiberwise surjective bundle homomorphisms $TM\to TX$. Moreover, the map g can be chosen arbitrarily C^0–close to f and with double folds contained in arbitrarily small balls.*

Remark 2.5 Recall that a special folded map $f\colon M\to X$ by definition has only spherical double folds, each fold component $C\subset M$ is a sphere whose image $\overline{C}\subset X$ is embedded and bounds a ball in X. In the equidimensional case ($d=0$) it is not possible, in general, to make images of fold components embedded. See Appendix A below.

Special folded mappings give a nice representation of (unstable) formal fibrations. As a class of maps, it turns out to be too small for our purposes. Namely, we wish to perform certain constructions (eg surgery) which does not preserve the class of special folded maps. Hence we consider a larger class of maps which allow for these constructions to be performed, and still small enough to admit a homotopically canonical extension to a formal fibration. This is the class of enriched folded maps.

2.3 Enriched folded maps and their suspensions

Recall that a folded map is, by definition, a map $f\colon M^{n+d}\to X^n$ which locally is of the form (13). In this section we study a certain extra structure on folded maps which we dub *framed membranes*.

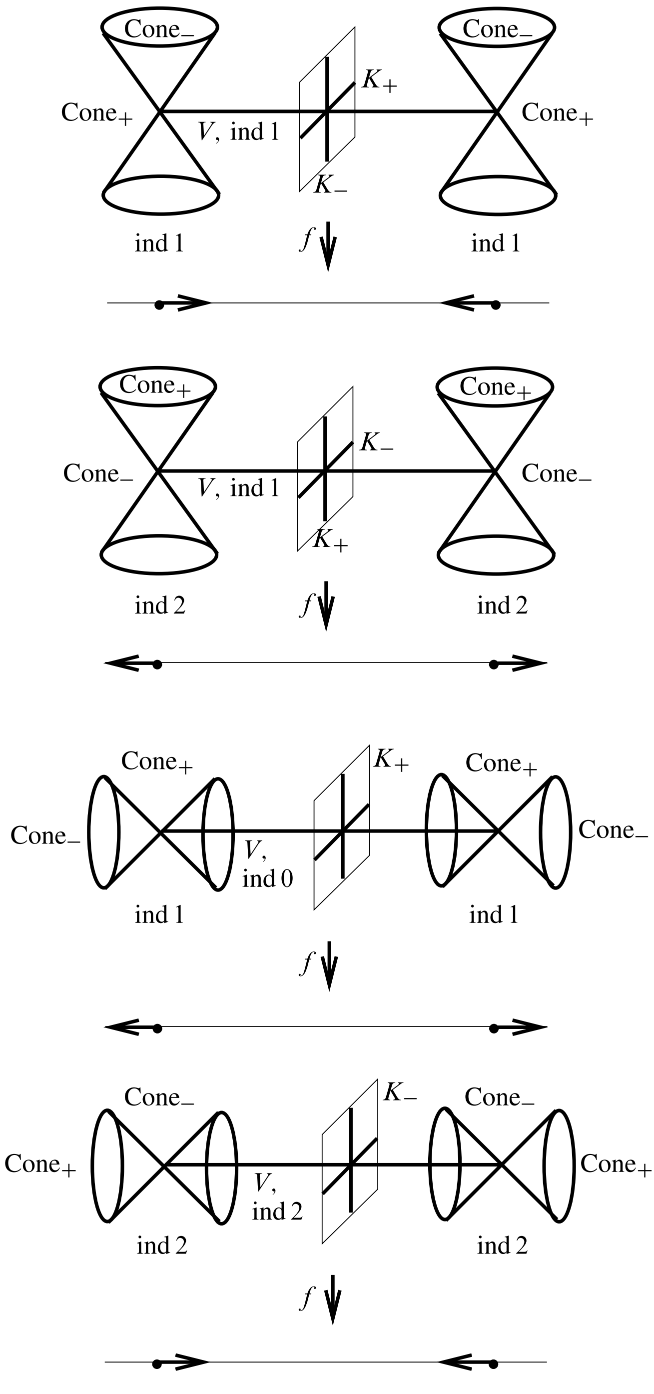


Figure 5: Pure membranes ($n = 1$, $d = 2$)

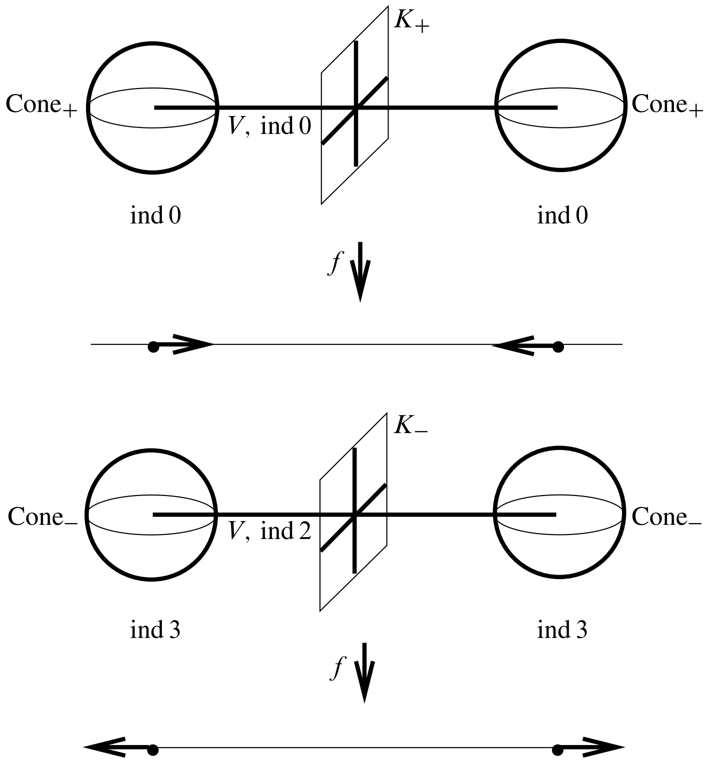


Figure 6: Pure membranes, continuation ($n=1$, $d=2$)

Definition 2.6 Let M^{n+d}, X^n be closed manifolds and $f\colon M \to X$ be a folded map. A *framed membrane* of index k, $k=0,\dots,d$, for f is a compact n–dimensional submanifold $V \subset M$ with boundary $\partial V = V \cap \Sigma(f)$, together with a *framing* $K=(K_-, K_+)$ where K_-, K_+ are *trivialized* subbundles of $(\operatorname{Ker} df)|_V$ of dimension k and $d-k$, respectively, such that

(i) the restrictions $f|_{\operatorname{Int} V}\colon \operatorname{Int} V \to X$ and $f|_{\partial V}\colon \partial V \to X$ are disjoint embeddings;

(ii) $K_\pm$ are transversal to each other and to TV;

(iii) there exists a coorientation of the image $\bar{C}$ of each fold component $C \subset \partial V$ such that $K_\pm|_C \subset \operatorname{Cone}_\pm(C)$;

Thus, over $\operatorname{Int} V$ we have $\operatorname{Ker} df = K_- \oplus K_+$, while over ∂V the bundle $\operatorname{Ker} df$ splits as $K_- \oplus K_+ \oplus \lambda$, where $\lambda = \lambda(C)$ is a line bundle contained in $\operatorname{Cone}_+(C) \cup \operatorname{Cone}_-(C)$. Note that the restriction $f|_V\colon V \to X$ is a topological embedding onto a smooth codimension 0 submanifold of X with boundary. However, f is not a smooth embedding because its rank drops at the boundary points.

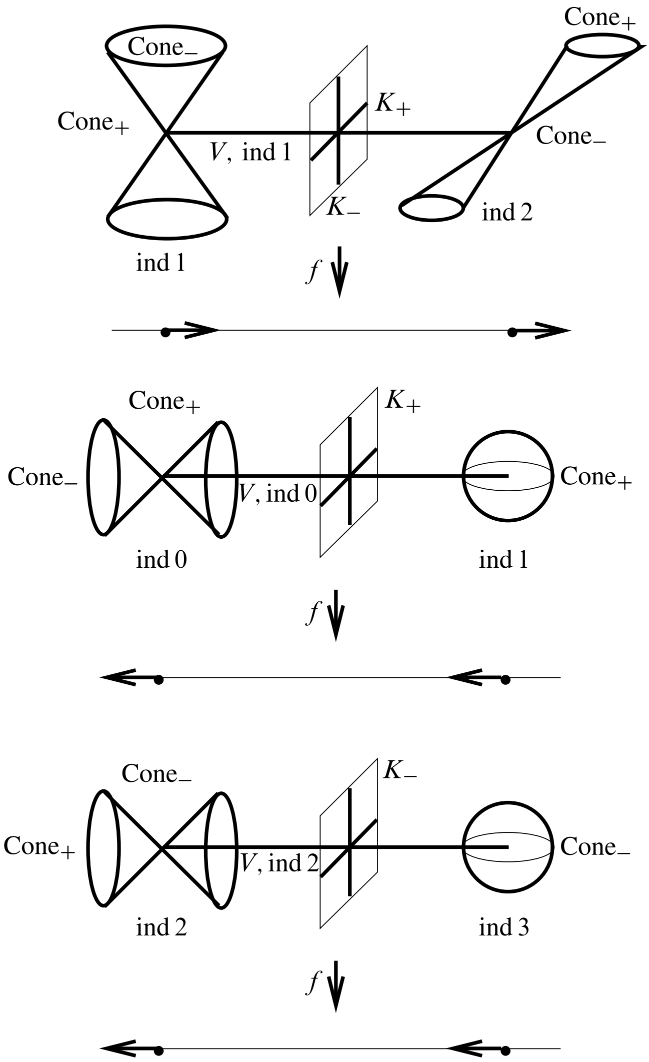


Figure 7: Mixed membranes ($n = 1$, $d = 2$)

A boundary component of a membrane V is called *positive* if $\lambda(C) \subset \mathrm{Cone}_+(C)$, and *negative* otherwise. The union of the positive boundary components will be denoted $\partial_+(V, K)$ and the union of the negative ones $\partial_-(V, K)$. Note that the coorientation of a component $\overline{C} \subset \partial \overline{V}$ implied by the definition of a framed membrane is given by inward normals to $\partial \overline{V}$ if $C \subset \partial_+(V, K)$, and by outward normals to $\partial \overline{V}$ if $C \subset \partial_-(V, K)$. The index of the fold component C is equal to k in the former case, and to $k + 1$ in the latter one.

We will call a framed membrane (V,K) *pure* if either $\partial_+(V,K)=\varnothing$ or $\partial_-(V,K)=\varnothing$. Otherwise we call it *mixed*. See Figures 5, 6 and 7.

Switching the roles of the subbundles K_+ and K_- gives a *dual* framing $\bar{K} = (\bar{K}_- = K_+, \bar{K}_+ = K_-)$. The index of the framed membrane $(V,\bar{K})$ equals $d-k$, and we also have $\partial_\pm(V,\bar{K}) = \partial_\mp(V,K)$.

Definition 2.7 An *enriched* folded map is a pair $(f,\mathfrak{e})$ where $f\colon M\to X$ is a folded map and $\mathfrak{e}$ is an *enrichment* of f, consisting of finitely many disjoint framed membranes $(V_1,K_1),\dots,(V_N,K_N)$ in M such that $\partial V = \Sigma(f)$, where V is the union of the V_i. If $n=\dim X>1$, we require in addition that $\partial_+ V_i$ is null homologous in X (ie the image of the fundamental class in $H_{n-1}(X)$ vanishes).

The last condition implies that $\partial_-\bar{V}_i$ is also null-homologous, since the membrane gives a cobordism between them. Let us also point out that the different V_i's are part of the structure (ie not just the union $V=\coprod V_i$). The V_i need not be connected, but f is injective on each V_i. The images $f(V_i)$ need not be disjoint.

Example 2.8 The double C–fold $w_C(n+d,n,k)$ defined by (15) is enriched folded in the following way. The annulus $A = C\times D^1\times 0\subset V\times\mathbb{R}\times\mathbb{R}^d$ is the membrane, and the trivialized bundles K_- and K_+ are defined as the span of

$$\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_k}\right)\quad\text{and}\quad\left(\frac{\partial}{\partial x_{k+1}},\dots,\frac{\partial}{\partial x_d}\right).$$

This is the *canonical enrichment* of the double C–fold $w_C(n+d,n,k)$. In particular, *any special folded map has a canonical enrichment.* Note that we have $\partial_+A = A\times\{-1\}$ and $\partial_-A = A\times\{1\}$.

From the tubular neighborhood theorem and a parametrized version of Morse's lemma, we get:

Lemma 2.9 *Let (V,K) be a connected framed membrane for an enriched folded map $(f,\mathfrak{e})$. Let $C=\partial V$ and let $\bar{C}=f(C)$ be its image (recall that we assumed $\bar{C}$ has no self-intersections). Then there is a tubular neighborhood $U_C\subset M$ of C with coordinate functions*

$$(y,u,x)\colon U_C\to C\times\mathbb{R}\times\mathbb{R}^d$$

and a tubular neighborhood $U_{\bar{C}}\subset X$ with coordinate functions

$$(y,t)\colon U_{\bar{C}}\to C\times\mathbb{R},$$

such that we have

- *$\partial/\partial u \in \lambda(C)$ along C;*
- *the vector field $\partial/\partial t$ defines the coorientation of $\bar{C}$ implied by the framing of V;*
- *the vector fields $\partial/\partial x_1, \dots, \partial/\partial x_k$, restricted to a neighborhood of C in V, belong to K_- and provide its given trivialization, while $\partial/\partial x_{k+1}, \dots, \partial/\partial x_d$ provide the given trivialization of K_+;*
- *in these local coordinates, f is given by*

$$f(y,u,x) = (y, t(x,u)),$$

where

$$t(x,u) = Q_k(x) \pm u^2 = -\sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{d} x_i^2 \pm u^2,$$

and $V \cap U_C$ coincides with $\{x = 0, \pm u \geq 0\}$, where the signs in the above formulas coincide with the sign of the boundary component C of the framed membrane (V, K).

A *suspension* of a folded map $f\colon M \to X$ is a surjective homomorphism $\Phi\colon TM \oplus \epsilon^1 \to f^*TX \oplus \epsilon^1$ such that $\pi_X \circ \Phi|_{TM} \circ i_M = df$, where π_X is the projection $TX \oplus \epsilon^1 \to TX$ and $i_M\colon TX \to TX \oplus 0 \hookrightarrow TX \oplus \epsilon^1$ is the inclusion. The following proposition is our main motivation for studying enrichments.

Proposition 2.10 *To an enriched folded map $(f, \mathfrak{e})$ we can associate a homotopically well defined suspension $\mathcal{L}(f, \mathfrak{e})$.*

Proof The suspension $TM \oplus \epsilon^1 \to f^*TX \oplus \epsilon^1$ will be of the form

$$\begin{pmatrix} df & v \\ \alpha & q \end{pmatrix}, \tag{16}$$

where $\alpha\colon TM \to \epsilon^1$ is a 1–form, v is a section of f^*TX, and q is a function.

The 1–form α is defined as $\alpha = du$ near ∂V, using the local coordinate u on U_C. To extend it to a 1–form on all of M, we will extend the function u. We first construct convenient local coordinates near V.

The map $f\colon M \to X$ restricts to a local diffeomorphism on $\mathrm{Int}(V)$. The local coordinate functions $x = (x_1, \dots, x_d)$ near ∂V from the normal form Lemma 2.9 extend to $\mathcal{O}p\, V$ in such a way that the vector fields $\partial/\partial x_1, \dots, \partial/\partial x_k$ along V generate the bundle K_-, while $\partial/\partial x_{k+1}, \dots, \partial/\partial x_k$ along V generate the bundle K_+. The tubular neighborhood theorem then gives a neighborhood $U_{\mathrm{Int}\, V} \subset M$ with coordinate functions

$$(z, x)\colon U_{\mathrm{Int}\, V} \to \mathrm{Int}\, V \times \mathbb{R}^d \tag{17}$$

satisfying $f(z,x)=f(z,0)$. After possibly reparametrizing in the x–coordinate, we may assume that the function $x\colon U_{\mathrm{Int}\,V}\to\mathbb{R}^d$ agrees with the function from Lemma 2.9 (their first derivatives already agree on $\mathrm{Int}\,V$). On the overlap $U_{\mathrm{Int}\,V}\cap U_C$, we can write the function $u\colon U_C\to\mathbb{R}$ from Lemma 2.9 in the local coordinates (17). Indeed, we have $f(z,x)=Q(x)\pm u(z,x)^2$, so

$$\pm u(z,x)^2 = f(z,x)-Q(x) = f(z,0)-Q(x) = \pm u(z,0)^2 - Q(x)$$

so
$$u(z,x)=\sqrt{u(z,0)^2\mp Q(x)}.$$

If we Taylor expand the square root, we get

$$u(z,x)=\gamma(z)\mp\delta(z)Q(x)+o(|x|^2). \tag{18}$$

for positive functions γ,δ. The function Q extends over $U_{\mathrm{Int}\,V}$ (use the same formula in the local coordinates of the tubular neighborhood of $\mathrm{Int}\,V$), and hence we can also extend u to a neighborhood of V inside $U_V=U_C\cup U_{\mathrm{Int}\,V}$, such that on $U_{\mathrm{Int}\,V}$ it satisfies (18). Extend u to all of M in any way, and let $\alpha=du$.

We have defined a bundle map $(df,\alpha)\colon TM\to f^*TX\oplus\epsilon^1$ which is surjective precisely when $\alpha|_{\mathrm{Ker}\,df}\neq 0$. Near V, $\alpha|_{\mathrm{Ker}\,df}=0$ precisely when $x=0\in\mathbb{R}^d$. It remains to define the section (v,q) of $f^*TX\oplus\epsilon^1$. Pick a function $\theta\colon U_V\to[-\pi,\pi]$ such that $u=-\sin\theta$ near ∂V, is negative on $\mathrm{Int}\,V$ and equal to $-\pi$ on $V-U_C$, and which is equal to π outside a small neighborhood of V. Then setting

$$v(u)=(\cos\theta)\frac{\partial}{\partial u} \tag{19}$$

$$q(u)=\sin\theta \tag{20}$$

completes the proof. □

Remark 2.11 Changing the sign of $v(u)$ in the formula (19) provides another suspension of the enriched folded map $(f,\mathfrak{e})$ which we will denote by $\mathcal{L}_-(f,\mathfrak{e})$. If n is even then the two suspensions $\mathcal{L}(f,\mathfrak{e})$ and $\mathcal{L}_-(f,\mathfrak{e})$ are homotopic.

Remark 2.12 Most (but not all) of the data of an enrichment $\mathfrak{e}$ of a folded map $f\colon M\to X$ can be reconstructed from the suspension $\Phi=\mathcal{L}(f,\mathfrak{e})$. If we write Φ in the matrix form (16), the manifold V is the set of points with $q\leq 0$ and $(df,\alpha)\colon TM\to f^*TX\oplus\epsilon^1$ not surjective. The partition of ∂V into $\partial_\pm(V,K)$ is determined by the coorientation of images of the fold components. On the other hand, the splitting $\mathrm{Vert}=K_+\oplus K_-$ cannot be reconstructed from the suspension.

Lemma 2.13 *Let $f\colon M \to X$ be a special folded map. Then the suspension $\mathcal{L}(f, \mathfrak{e})$ (as well as the suspension $\mathcal{L}_-(f, \mathfrak{e})$) of the canonically enriched folded map $(f, \mathfrak{e})$ is homotopic to its stabilized regularized differential $\mathcal{R}df$.*

Proof This can be seen in the local models. □

2.4 Destabilization

So far (in Theorem 2.4 and Lemma 2.13) we have related *unstable* formal fibrations to (enriched) folded maps. In Theorem 1.8 we need to work with *stable* formal fibrations (because that is what $\Omega^\infty \mathbb{C}P^\infty_{-1}$ classifies). In this section we study the question of whether an epimorphism $\Phi\colon TM \oplus \epsilon^1 \to TX \oplus \epsilon^1$ can be "destabilized", ie homotoped to be of the form $\bar{\Phi} \oplus \mathrm{Id}$ for some unstable epimorphism $\bar{\Phi}\colon TM \to TX$. This is not possible in general of course (the obstruction is an Euler class). Instead we prove the following.

Proposition 2.14 *Let $\Phi\colon TM \oplus \epsilon^1 \to TX \oplus \epsilon^1$ be a bundle epimorphism with underlying map $f\colon M \to X$. Assume M and X are connected. Then there is a compact codimension 0 submanifold $M_0 \subset M$ which is homotopy equivalent to a simplicial complex of dimension at most 1, such that the following properties hold, after changing f and Φ by a homotopy (in the class of bundle epimorphisms).*

- (i) *$f|_{M_0}$ is folded and has an enrichment $\mathfrak{e}$ such that*
  - *$\Phi|_{M_0} = \mathcal{L}(f_{|M_0}, \mathfrak{e})$ if $n = \dim X > 1$;*
  - *$\Phi_{|M_0} = \mathcal{L}(f|_{M_0}, \mathfrak{e})$ or $\Phi|_{M_0} = \mathcal{L}_-(f_{|M_0}, \mathfrak{e})$ in the case $n = 1$;*
- (ii) *Φ is integrable near ∂M_0, ie it equals $Df \oplus \epsilon^1$ there.*
- (iii) *Φ destabilizes outside M_0, ie it equals $\bar{\Phi} \oplus \epsilon^1$ there, for some bundle epimorphism $\bar{\Phi}\colon TM|_{M \setminus M_0} \to TX$.*

The strategy of the proof of the proposition is as follows. First we forget about (iii) in the proposition, and only worry about how f and Φ look like on M_0. We prove that this can be done for a large class of possible M_0's. After that, we consider the obstruction to destabilizing Φ outside M_0 without changing it on M_0. This obstruction is essentially an integer, and we prove that M_0 can be chosen so that the obstruction vanishes.

We first give a local model for the enriched folded map $M_0 \to X$. Let $I = [-1, 1]$ be the interval, and let $K \subset \mathrm{Int}(I^n)$ be a simplicial complex. We will consider $I^n \subset I^{n+d}$ as the subset $I^n \times \{0\}$. Let $U_0 \subset I^n$ be a regular neighborhood of $K \subset I^n$, and let $U \subset I^{n+d}$ be a regular neighborhood of $K \subset I^{n+d}$. Let $\pi\colon I^{n+d} \to I^n$ be the projection. In order to avoid confusion we will write $\bar{U}_0 = U_0 \times \{0\} \subset I^{n+d}$, and hence $U_0 = \pi(\bar{U}_0)$. We can assume that $\pi_{|\partial U}\colon \partial U \to I^n$ is a folded map, with fold $\partial\bar{U}_0 \subset \partial U$ which has index 0 with respect to the inward coorientation of $\partial U_0 \subset U_0$.

Let $N = \partial U \times [-1,1]$ be a bicollar of ∂U, ie an embedding $N \to I^{n+d}$, which maps $(u,0) \mapsto u \in \partial U$ and $\partial U \times [-1,0]$ to U. Let $M_0 = U \cup N$. We construct a folded map $M_0 \to M_0$ in the following way. First pick a map $\varphi = (\varphi_1,\varphi_2)\colon [-1,1] \to [-1,1] \times \mathbb{R}$ with the following properties.

(i) $\varphi(\pm s) = (-s,0)$ for $s > 1/2$;

(ii) $\varphi_1'(s) > 0$ for $s < 0$, $\varphi_1'(s) < 0$ for $s > 0$, and $\varphi_1''(0) < 0$;

(iii) $\varphi_2'(0) < 0$.

In particular φ is an immersion of codimension 1, and $\varphi_1\colon [-1,1] \to [-1,1]$ is a folded map with fold $\{0\}$. Extend to an immersion $\varphi\colon [-1,1] \times \mathbb{R} \to [-1,1] \times \mathbb{R}$ with the property that

$$\varphi(\pm s,t) = (-s,\pm t)$$

for $s > 1/2$. Recall that $N = \partial U \times [-1,1]$ and construct a codimension 0 immersion $\gamma_0\colon N \times \mathbb{R} \to N \times \mathbb{R}$ by

$$\gamma_0(u,s,t) = (u,\varphi(s,t)),$$

for $u \in \partial U$. Extend to a codimension 0 immersion $\gamma_1\colon M_0 \times \mathbb{R} \to M_0 \times \mathbb{R}$ by

$$\gamma_1(x,t) = (x,-t)$$

for $x \in M_0 \setminus N$. For $m \in M_0$, let $\gamma_1(m) \in M_0$ be the first coordinate of $\gamma_1(m,0) \in M_0 \times \mathbb{R}$. Differentiating γ_1 then gives a bundle map

$$\Gamma\colon TM_0 \oplus \epsilon^1 \to TM_0 \oplus \epsilon^1 \tag{21}$$

with underlying map $\gamma\colon M_0 \to M_0$. We record some of its properties in a lemma.

Lemma 2.15 (i) *The map γ_1 is homotopic in the class of submersions (=immersions) to the map* $\mathrm{Id} \times (-1)\colon M_0 \times \mathbb{R} \to M_0 \times \mathbb{R}$.

(ii) *Let $M_0 \subset I^{n+d}$ and γ and Γ be as above. Then γ is a folded map. The fold is $\partial U \subset N \subset M_0$, and the image of the fold is also ∂U. Near ∂M_0, the bundle map Γ is* integrable, *ie* $\Gamma = D\gamma \oplus \epsilon^1$.

(iii) *Let $\pi\colon I^{n+d} \to I^n$ be the projection and define a bundle map*

$$H\colon TM_0 \oplus \epsilon^1 \to T(I^n) \oplus \epsilon^1$$

by $H = (D\pi \oplus \epsilon^1) \circ \Gamma$. It covers the folded map $h = \pi \circ \gamma\colon M_0 \to I^n$, which has fold $\partial \bar{U}_0 \subset \partial U \subset M_0$. The image of the fold is $\partial U_0 \subset I^n$, and it has index 0 with respect to the inward coorientation of U_0.

(iv) *A membrane for the underlying map h can be defined as $V = \bar{U}_0$ with the framing $K = (K_+ = \mathrm{Span}(\partial/\partial x_{n+1}, \dots, \partial/\partial x_{n+d}$, and $\mathrm{Ker}_-(V)), K_- = \{0\}$. This defines an enrichment $\mathfrak{e}$ for h. Finally, H is integrable over ∂M_0 and $H = \mathcal{L}(h, \mathfrak{e})$.*

Proof We leave this as an easy exercise for the reader. See also Section 3.2.3 and the proof of Proposition 2.10. □

Let us also point out that we could equally well have based the construction on a map $\varphi_-\colon [-1,1] \to [-1,1] \times \mathbb{R}$ defined as φ above, except that we replace the condition $\varphi_2'(0) < 0$ by $\varphi_2'(0) > 0$. Using this map as a basis for the construction gives a bundle epimorphism

$$H_-\colon TM_0 \oplus \epsilon^1 \to T(I^n) \oplus \epsilon^1$$

which also satisfies the conclusion of Lemma 2.15, except that (iv) gets replaced by $H = \mathcal{L}_-(h, \mathfrak{e})$.

Proof of Proposition 2.14 (i) and (ii) Let $\Phi\colon TM \oplus \epsilon^1 \to TX \oplus \epsilon^1$ be as in the proposition. We will prove that given *any* $M_0 \subset M$ which is a regular neighborhood in M of a simplicial complex $K \subset I^n \times \{0\} \subset I^{n+d} \subset M$, there is a homotopy of f and Φ in the class of bundle epimorphisms, after which (i) and (ii) hold.

By Phillips' theorem [12] we can assume that Φ is induced by a submersion

$$\Psi\colon M \times \mathbb{R} \to X \times \mathbb{R}.$$

Pick cubes $D = I^{n+d} \subset M$ and $I^n \subset X$. We regard $M_0 \subset D \subset M$ and let $\pi\colon D \to I^n$ denote the projection to the first n coordinates. We can assume that $\Psi(D \times \mathbb{R}) \subset I^n \times \mathbb{R}$. The space of submersions $D \times \mathbb{R} \to I^n \times \mathbb{R}$ is homotopy equivalent to $O(n+d+1)/O(d)$ which is connected, and hence we may assume that $\Psi_{|D} = \pi \times (-1)$. (Here we used $d \geq 1$. In the case $d = 0$, we get $O(n+d+1)$ which has two path components, but after possibly permuting coordinates on $D = I^{n+d}$, we may assume that $\Psi_{|D}$ is in the same path component as $\pi \times (-1)$.) Hence, after a homotopy of Ψ in the class of submersions, we may assume that $\Psi_{|M_0} = \pi \times (-1)$.

By Lemma 2.15(i), we can assume that Ψ agrees with the map $(\pi \times \mathbb{R}) \circ \gamma_1$, possibly after a further deformation of Ψ in a neighborhood of $M_0 \subset D$. This proves (i) and (ii) in the proposition. □

It remains to prove that for a suitably chosen of K (and hence M_0), we can destabilize Φ outside M_0. There is an obstruction to doing this, which is essentially an integer, depending on the homotopy class of Φ and on the Euler characteristic of K. So far, the simplicial complex K could be arbitrary, the only restriction being that it embeds into I^n. Here, $n = \dim X > 1$, so K can be any 1–dimensional simplicial complex and in particular $\chi(K)$ can be any integer. Using this, we pick a K for which the obstruction to destabilizing outside M_0 vanishes. Let us explain these ideas more explicitly.

Let s denote a section of $TM \oplus \epsilon^1$ such that $\Phi \circ s$ is the constant section $(0,1) \in f^*(TX) \oplus \epsilon^1$. This defines s uniquely up to homotopy (in fact s is unique up to translation by vectors in the kernel of Φ). Over M_0 the epimorphism Φ is induced by a composition

$$M_0 \times \mathbb{R} \xrightarrow{\gamma_1} M_0 \times \mathbb{R} \xrightarrow{\text{proj}} X \times \mathbb{R},$$

and on M_0 we may choose s so that $D\gamma_1$ takes s to a unit vector in the $\mathbb{R}$–direction. Another relevant section is the constant section $s_\infty(x) = (0,1) \in S(TM \oplus \epsilon^1)$. We have $s(x) = s_\infty(x)$ for $x \in \partial M_0$. Our aim is to change Φ by a homotopy and achieve $s(x) = s_\infty(x)$ for all x outside M_0. In each fiber, $s(x) \in S(T_x M \oplus \mathbb{R}) = S^{n+d}$, so by induction on cells of $M \setminus D$, we can assume that $s(x) = s_\infty(x)$ outside D, since $M \setminus D$ can be built using cells of dimension at most $n+d-1$. It remains to consider $D \setminus M_0$. Let s_K be the section which agrees with s on M_0 and with s_∞ outside M_0. Thus s and s_K are both sections of $S(TM \oplus \epsilon^1)$ which equal s_∞ outside D. We study their homotopy classes in the space of such sections.

Using stereographic projection, the fiber of $S(TM \oplus \epsilon^1)$ at a point $x \in M$ can be identified with the one-point compactification of $T_x M$. Hence we can think of sections as continuous vector fields on M, which are allowed to be infinite. The section at infinity is $s_\infty(x) = (0,1) \in S(T_x M \oplus \mathbb{R})$. In this picture we have the following way of thinking of s_K: For $x \in \partial U$, $s_K(x)$ is a unit vector orthogonal to ∂U pointing outwards. Moving x away from ∂U to the *inside* makes $s_K(x)$ smaller and it gets zero as we get far away from ∂U. Moving x away from ∂U to the *outside* makes $s_K(x)$ larger and it gets infinite as we get far away from ∂U. The section s_K depends up to homotopy only on the simplicial complex $K \subset I^n$, hence the notation.

Lemma 2.16 *There is a bijection between $\mathbb{Z}$ and sections of $S(TM \oplus \epsilon^1)$ which agree with s_∞ outside D. The bijection takes s_K to $\chi(K) \in \mathbb{Z}$.*

Proof The tangent bundle TM is trivial over D, so the space of such sections is just the space of pointed maps $S^{n+d} \to S^{n+d}$ and homotopy classes of such are classified by their degree, which is an integer.

We have assumed $U \subset D = I^{n+d}$ so using the standard embedding $D \subset \mathbb{R}^{n+d}$ we can work entirely inside $\mathbb{R}^{n+d}$. The geometric interpretation of s_K given above can then be rephrased even more conveniently. Let $r\colon U \to K$ be the retraction in the tubular neighborhood, and let

$$\widetilde{s}_K(x) = x - r(x) \tag{22}$$

for $x \in U$. Pick any continuous extension of $\widetilde{s}_K$ to D with the property that when $x \notin U$, we have

$$\widetilde{s}_K(x) \in (T_x M - \{0\}) \cup \{\infty\}.$$

Up to homotopy there is a unique such extension because we are picking a point $\widetilde{s}_K(x)$ in a contractible space. Then $\widetilde{s}_K \simeq s_K$.

To calculate the degree of the corresponding map we perturb even further. Remember that any simplicial complex K has a standard vector field with the following property: The stationary points are the barycenters of simplices and the flowline starting at a point x converges to the barycenter of the open cell containing x. Let $\psi_\epsilon\colon K \to K$ be the time ϵ flow of this vector field, and define a vector field $\widehat{s}_K$ just like $\widetilde{s}_K$, except that we replace the right hand side of (22) by $x - \psi_\epsilon \circ r(x)$ for some small $\epsilon > 0$.

The resulting vector field vanishes precisely at the barycenters of K, and the index of the vector field at the barycenter of a simplex σ is $(-1)^{\dim(\sigma)}$. The claim now follows from the Poincaré–Hopf theorem. □

Proof of Proposition 2.14 (iii) Let us first consider the case $n > 1$. We have proved that all possible sections of $S(TM \oplus \epsilon^1)$ which agree with s_∞ outside D, are homotopic to s_K for some K. Then we can choose K such that $s_K \simeq s$. Since s_K and s agree on M_0 there is a homotopy of s, fixed over M_0, so that $s(x) = s_\infty(x)$ for all $x \in M - M_0$. This homotopy lifts to a homotopy of bundle epimorphisms $\Phi\colon TM \oplus \epsilon^1 \to TX \oplus \epsilon^1$, and then (iii) is satisfied.

For $n = 1$ we may not be able to choose a $K \subset X$ with the required Euler characteristic, since subcomplexes of 1–manifolds always have nonnegative Euler characteristic. However, vector fields of negative index can be achieved as $-s_K$, and that is the vector field that arises if we use the negative suspension $\mathcal{L}_-(f, \mathfrak{e})$. □

2.5 From formal epimorphisms to enriched folded maps

The following theorem summarizes the results of Section 2.

Theorem 2.17 *Let $\Phi\colon TM \oplus \epsilon^1 \to TX \oplus \epsilon^1$ be a bundle epimorphism and write $n = \dim X$ and $d = \dim M - \dim X$. Suppose that $d > 0$ and Φ is integrable in a neighborhood of a closed set $A \subset M$ (when X and M has boundary, we assume $\varphi^{-1}(\partial X) = \partial M \subset A$). Then there is a homotopy of epimorphisms $\Phi_t\colon TM \oplus \epsilon^1 \to TX \oplus \epsilon^1$, $t \in [0,1]$, fixed near A, which covers a homotopy $\varphi_t\colon M \to X$, such that $\varphi_1\colon M \to X$ is folded, and $\Phi_1 = \mathcal{L}(\varphi_1, \mathfrak{e})$ for some enrichment $\mathfrak{e}$ of φ_1. If $n > 1$ then the image $\bar{C} \subset X$ of each fold component $C \subset M$ of φ_1 bounds a domain in X.*

Proof First use Proposition 2.14 to make φ enriched folded over a domain M_0, such that Φ destabilizes outside M_0. Then use Theorem 2.4 and Lemma 2.13 to make (φ, Φ) special enriched outside M_0. □

3 Beginning of the proof of Theorem 1.8

With the exception of Section 3.2 we shall assume from now on that $d = 2$.

3.1 Cobordisms of folded maps

Let us rephrase the results of the previous section more systematically, and put them in the context of the overall goal of the paper. So far we have mainly studied the relation between formal fibrations and enriched folded maps. Let us formalize the result. We consider various bordism categories of maps $f\colon M \to X$ such that M and X are both oriented and X is a compact manifold, possibly with boundary, and which satisfy the following two conditions:

(C1) f has T_∞ ends, ie there is (as part of the structure) a germ at infinity of a diffeomorphism $j\colon T_\infty \times X \rightsquigarrow M$ such that $f \circ j = \pi$, where π is a germ at infinity of the projection $X \times T_\infty \to X$. This trivialized end will be called the *standard end* of M.

(C2) There is a neighborhood U of ∂X such that $f^{-1}(U) \to U$ is a fibration (ie smooth fiber bundle) with fiber T_∞.

In all the categories introduced below, morphisms are maps $F\colon W \to Y$ to oriented cobordisms Y between manifolds $X_\pm = \partial_\pm Y$. The manifolds $X_\pm$ are allowed to have nonempty boundary, and in this case the cobordism is required to be trivial and trivialized over the boundary, ie Y is an oriented manifold with boundary $\partial Y = \partial_- Y \cup \partial_+ Y$ satisfying $\partial_- Y \cap \partial_+ Y = \partial X_+ = \partial X_-$

Definition 3.1 (i) Fib is the category of *fibrations* (smooth fiber bundles) with fiber T_∞, which satisfy (C1) and (C2).

(ii) $\mathsf{Fold}^\$$ is the category of enriched folded maps, satisfying (C1) and (C2).

(iii) FFib is the category of *formal fibrations*, ie bundle epimorphisms $\Phi\colon TM\oplus\epsilon^1 \to TX\oplus\epsilon^1$ with underlying map $f\colon M\to X$, such that f satisfies (C1) and (C2), and such that $\Phi = df\oplus\epsilon^1$ near $f^{-1}(\partial X)$.

We have functors

$$\mathsf{Fib}\to\mathsf{Fold}^\$ \xrightarrow{\mathcal{L}} \mathsf{FFib}.$$

The functor $\mathsf{Fib}\to\mathsf{Fold}^\$$ is the obvious inclusion. Everything we said in Section 2 works just as well with the conditions (C1) and (C2) imposed, and hence Proposition 2.10 gives the functor $\mathcal{L}\colon \mathsf{Fold}^\$ \to \mathsf{FFib}$.

In this setup, the main goal of the paper is to prove that any object in FFib is cobordant to one in Fib. Note that Theorem 2.17 is formulated as an extension result, and hence it is applicable to manifolds with boundary and implies the following corollary.

Corollary 3.2 *Any object in* FFib *is cobordant in* FFib *to an object in the image of the functor* $\mathcal{L}$.

It remains to see that any object of $\mathsf{Fold}^\$$ is cobordant to one in Fib. In fact Theorem 2.17 is a little stronger: any object of FFib is cobordant to one in the image of $\mathcal{L}$ using only homotopies of the underlying maps, ie no cobordisms of M and X. In contrast, comparing Fib to $\mathsf{Fold}^\$$ involves changing M and X by *surgery*. The surgery uses the membranes in the enrichment, and also makes crucial use of a form of Harer's stability theorem.

In fact, it is convenient to work with a slight modification of the category $\mathsf{Fold}^\$$.

Definition 3.3 Let $\widetilde{\mathsf{Fold}}^\$$ be the category with the same objects as $\mathsf{Fold}^\$$, but where we formally add morphisms which create double folds along a submanifold C, ie singularities of the form (15). More precisely, by Example 2.8, the double C fold has a canonical enrichment, and we formally add morphisms in $\widetilde{\mathsf{Fold}}^\$$ in both directions between the map $f\colon M\to X$ and the same map $f'\colon M\to X$ with a double C–fold singularity created. When $n>1$ we require in addition that C be homologically trivial.

Remark 3.4 It turns out that when $n>1$ there is, in fact, no real difference between the categories $\mathsf{Fold}^\$$ and $\widetilde{\mathsf{Fold}}^\$$: if two objects in $\widetilde{\mathsf{Fold}}^\$$ are cobordant, then they are already cobordant in $\mathsf{Fold}^\$$. We shall not need this fact and leave its proof to the reader as an exercise.

According to Lemma 2.13, adding a double C–fold together with its canonical enrichment changes $\mathcal{L}(f,\mathfrak{e})$ only by a homotopy. Hence the functor $\mathcal{L}\colon \mathsf{Fold}^{\$} \to \mathsf{FFib}$ extends to a functor $\widetilde{\mathcal{L}}\colon \widetilde{\mathsf{Fold}}^{\$} \to \mathsf{FFib}$. In the proof of our main theorem, $\mathsf{Fold}^{\$}$ is just a middle step, and it turns out to be more convenient to work with the modified category $\widetilde{\mathsf{Fold}}^{\$}$.

We prove that any object of $\widetilde{\mathsf{Fold}}^{\$}$ is cobordant to one in Fib in two steps.

Definition 3.5 Let $\mathsf{Fold}_h^{\$} \subset \mathsf{Fold}^{\$}$ be the subcategory where objects and morphisms are required to satisfy the following conditions.

(i) Folds are hyperbolic, ie have no folds of index 0 and 3.

(ii) For all $x \in X$, the manifold $f^{-1}(x) - \Sigma(f)$, ie the fiber minus its singularities, is connected.

Let $\widetilde{\mathsf{Fold}}_h^{\$}$ be the category with the same objects, but where we formally add morphisms (in both directions) which create double folds.

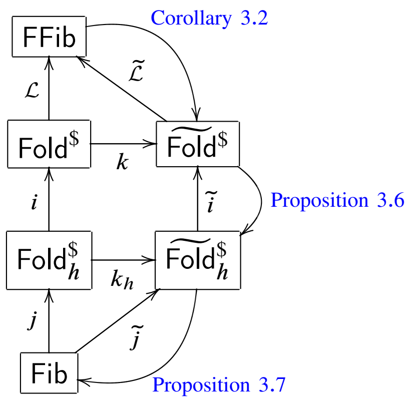
Our main result, Theorem 1.8, follows from Corollary 3.2 and the following two propositions.

Proposition 3.6 *Any enriched folded map* $(f\colon M \to X, \mathfrak{e}) \in \mathsf{Fold}^{\$}$ *is bordant in the category* $\widetilde{\mathsf{Fold}}^{\$}$ *to an element in* $\mathsf{Fold}_h^{\$}$.

Proposition 3.7 *Any enriched folded map* $(f\colon M \to X, \mathfrak{e}) \in \mathsf{Fold}_h^{\$}$ *is bordant in the category* $\widetilde{\mathsf{Fold}}_h^{\$}$ *to a fibration from* $\mathsf{Fib} \subset \mathsf{Fold}^{\$}$.

The proof of the latter uses Harer's stability Theorem 1.9. This is the only part of the whole story in which the condition $d = 2$ is used in an essential way.

The next diagram summarizes all the categories we introduced, as well as relations among them. Here $i, \widetilde{i}, j, \widetilde{j}, k, k_h$ are inclusion maps.



3.2 Fold surgery

In this section we develop a technique of surgery of fold singularities which will be an essential tool in our proof of Propositions 3.6 and 3.7. We shall only need the case $d=2$, but it is natural to describe the surgery technique for a general $d \geq 0$.

Let $f\colon M \to X$ be a folded map with cooriented folds. Let $C \subset \Sigma(f)$ be a connected component of index k, and let $\overline{C} \subset X$ be its image. For $p \in \overline{C}$, the fiber $f^{-1}(p)$ has a singularity. There are two directions in which we can move p away from $\overline{C}$ to resolve the singularity and get a manifold. The manifold we get by moving p to the positive side (with respect to the coorientation) differs from the manifold we get by moving p to the negative side by a surgery of index k, ie it has an embedded $D^k \times \partial D^{d-k}$ instead of a $\partial D^k \times D^{d-k}$. If $\overline{C}$ bounds an embedded domain $\overline{P} \subset X$, then one can try to prevent the surgery from happening by performing an inverse Morse surgery fiberwise in each fiber $f^{-1}(p)$, $p \in \overline{P}$. This process is called *fold eliminating surgery* and we proceed to describe it in more detail.

3.2.1 Surgery template We begin with a local model for the surgery. Let P be an n–dimensional oriented manifold with collared boundary ∂P. The collar consists of an embedding $\partial P \times [-1,0] \to P$, mapping $(p,0) \mapsto p$. Extend P by a bicollar $U = \partial P \times [-1,1]$, and set

$$\widetilde{P} = P \underset{\partial P \times [-1,0]}{\cup} U. \tag{23}$$

Let Q be a quadratic form of index k on $\mathbb{R}^{d+1}$:

$$Q(x) = -\|x_-\|^2 + \|x_+\|^2,$$

where $x = (x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1}$, $x_- = (x_1, \dots, x_k)$, $x_+ = (x_{k+1}, \dots, x_{d+1})$. Let $H \subset \mathbb{R}^{d+1}$ be the domain (see Figure 8)

$$H = \{|Q| \leq 1,\ \|x_+\| \leq 2\}.$$

We are going to use the map $Q\colon H \to [-1,1]$ as a prototype of a fold. The boundary of the (possibly singular) fiber $Q^{-1}(t)$ can be identified with $S^{k-1} \times S^{d-k-1}$ via the diffeomorphism

$$\{Q = t, \|x_+\| = 2\} \to S^{k-1} \times S^{d-k-1} \tag{24}$$

$$(x_-, x_+) \mapsto \left(\frac{x_-}{\|x_-\|}, \frac{x_+}{\|x_+\|} \right). \tag{25}$$

The map $\mathrm{Id} \times Q\colon \widetilde{P} \times H \to \widetilde{P} \times [-1,1]$ is a folded map with fold $P \times 0 \subset P \times H$ which has index k with respect to the coorientation of the fold defined by the second coordinate of the product $\widetilde{P} \times [-1,1]$.

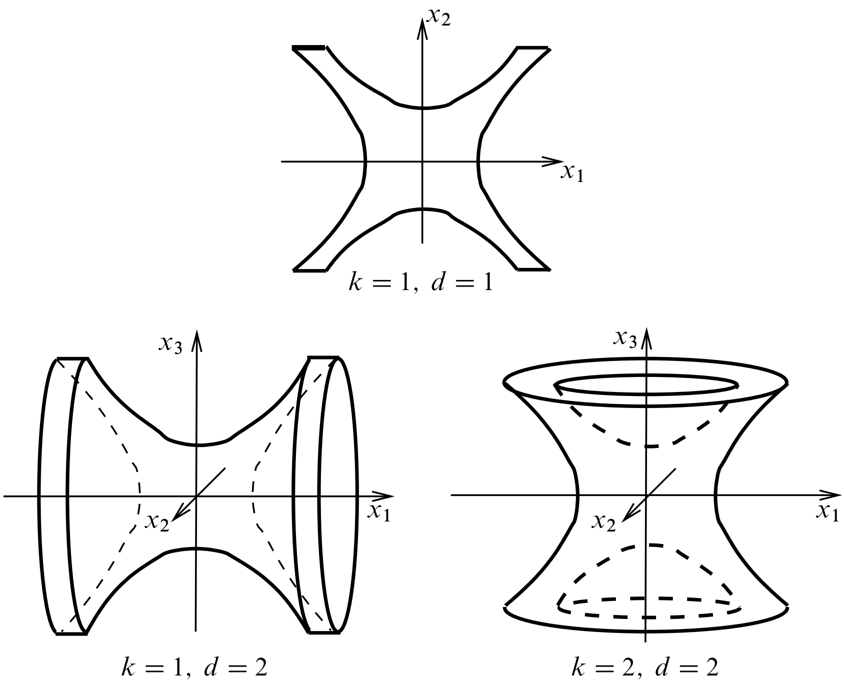


Figure 8: The domain H

Given a smooth function $\varphi\colon \widetilde{P} \to [-1,1]$ we define (see Figure 9)

$$(26)\qquad \begin{aligned} \widetilde{P}^{\varphi} &= \{(p,x) \in \widetilde{P} \times H \mid Q(x) = \varphi(p)\},\\ Z^{\varphi} &= \{(p,x) \in \partial\widetilde{P} \times H \mid Q(x) = \varphi(p)\},\\ R^{\varphi} &= \widetilde{P}^{\varphi} \cap \{\|x_+\| = 2\}. \end{aligned}$$

We then have $\partial\widetilde{P}^{\varphi} = Z^{\varphi} \cup R^{\varphi}$.

The restriction $\pi\colon \widetilde{P}^{\varphi} \to \widetilde{P}$ is our "template" for the result of the surgery. The next lemma records its properties.

Lemma 3.8 *Suppose that* 0 *is not a critical value of* φ. *Then* $\widetilde{P}^{\varphi}$ *is a smooth manifold of dimension* $n+d$, *and the projection* $\pi|_{\widetilde{P}^{\varphi}}\colon \widetilde{P}^{\varphi} \to \widetilde{P}$ *is a folded map with fold* $C = \widetilde{P}^{\varphi} \cap (\widetilde{P} \times \{0\})$, *which projects to* $\overline{C} = \pi(C) = \varphi^{-1}(0) \subset \widetilde{P}$. *In particular, the map* $\pi|_{\widetilde{P}^{\varphi}}$ *is nonsingular if* φ *does not take the value* 0. *The fold* C *has index* k *with respect to the coorientation of* $\overline{C}$ *by an outward normal vector field to the domain* $\{\varphi \le 0\}$.

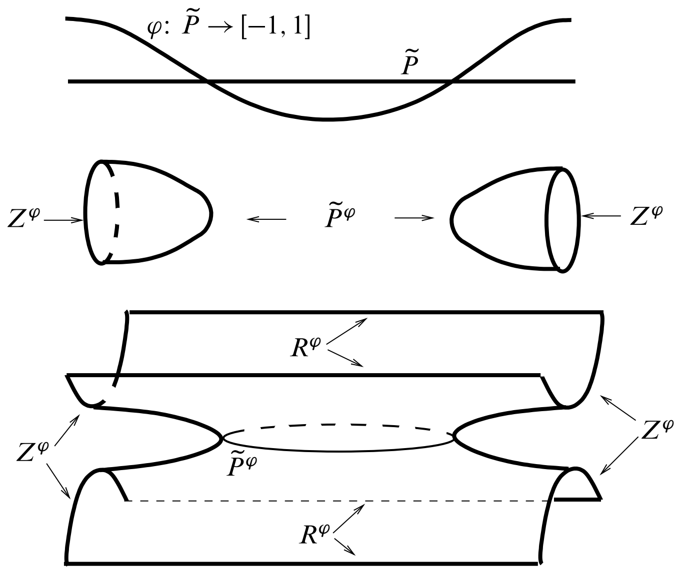


Figure 9: The manifold $\widetilde{P}^{\varphi}$ for $k=0, d=1, n=1$ and $k=1, d=1, n=1$

Given a one-parameter family of functions $\varphi_t\colon \widetilde{P} \to [-1,1]$, $t \in [0,1]$, we denote (see Figure 10)

$$\begin{aligned} \widetilde{P}^{\varphi_t} &= \{(p,x,t) \in \widetilde{P} \times H \times [0,1] \mid Q(x) = \varphi_t(p)\}, \\ Z^{\varphi_t} &= \{(p,x,t) \in \partial\widetilde{P} \times H \times [0,1] \mid Q(x) = \varphi_t(p)\}, \qquad (27) \\ R^{\varphi_t} &= P^{\varphi_t} \cap \{\|x_+\| = 2\}. \end{aligned}$$

We have $\partial \widetilde{P}^{\varphi_t} = Z^{\varphi_t} \cup R^{\varphi_t} \cap \widetilde{P}^{\varphi_0} \cup \widetilde{P}^{\varphi_1}$. We will consider the projection

$$\pi\colon \widetilde{P} \times H \times [0,1] \to \widetilde{P} \times [0,1]$$

and especially its restriction to the subsets (27). Using (24), the set R^{φ_t} can be identified with $(\widetilde{P} \times [0,1]) \times S^{k-1} \times S^{d-k-1}$ via a diffeomorphism over $\widetilde{P} \times [0,1]$. In particular we get a diffeomorphism

$$R^{\varphi_0} \times [0,1] \to R^{\varphi_t}, \qquad (28)$$

which scales the x_- coordinates. In fact it can be seen to be given by the formula

$$(p,(x_-,x_+),t) \mapsto \left(p, \left(\sqrt{\frac{4-\varphi_t(p)}{4-\varphi_0(p)}}x_-, x_+\right), t\right),$$

although we shall not need this explicit formula.

The restriction $\pi\colon \widetilde{P}^{\varphi_t} \to \widetilde{P} \times [0,1]$ is our "template cobordism". The next lemma records its properties.

Lemma 3.9 *Let $\varphi_t\colon \widetilde{P} \to [-1,1]$, $t \in [0,1]$, be a one-parameter family of functions such that 0 is not a critical value of φ_0 or φ_1 or of the function $\widetilde{P} \times [0,1] \to [-1,1]$ defined by $(p,t) \mapsto \varphi_t(p)$, $p \in \widetilde{P}$, $t \in [0,1]$. We also assume $\varphi_t(p)$ is independent of t for p near ∂P. Then $\pi|_{\widetilde{P}^{\varphi_t}}\colon \widetilde{P}^{\varphi_t} \to \widetilde{P} \times [0,1]$ is a folded cobordism between the folded maps $\widetilde{P}^{\varphi_0} \to \widetilde{P}$ and $\widetilde{P}^{\varphi_1} \to \widetilde{P}$. We have $Z^{\varphi_t} = Z^{\varphi_0} \times [0,1]$, so together with (28) we get a diffeomorphism*

$$(Z^{\varphi_0} \cup R^{\varphi_0}) \times [0,1] \to Z^{\varphi_t} \cup R^{\varphi_t}. \tag{29}$$

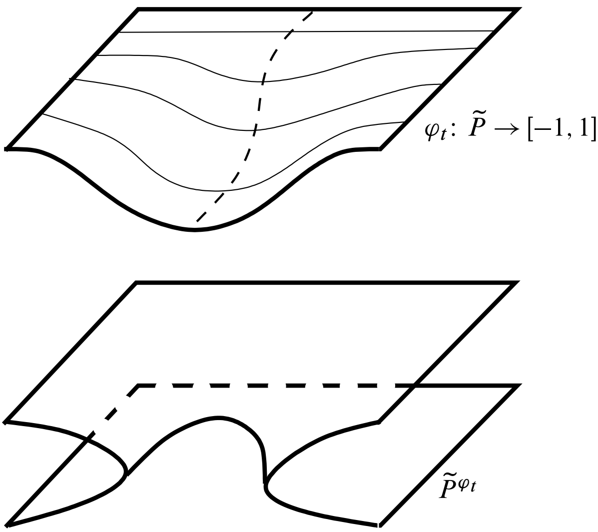


Figure 10: The functions φ_t and the cobordism $\widetilde{P}^{\varphi_t}$ for $k=0, d=0, n=1$

We will need to apply the above lemma to two particular functions on $\widetilde{P}$. Recall that $U = \partial P \times [-1,1] \subset \widetilde{P}$ denotes the bicollar. Take $\varphi_0 \equiv 1$ and pick a function φ_1 with the following properties:

- $\varphi_1 = 1$ near $\partial \widetilde{P}$,
- $\varphi_1 = -1$ on $\widetilde{P} \setminus U$,
- For $(p,v) \in U$, $\varphi_1(p,v)$ is a nondecreasing function of v,
- $\varphi_1(p,v) = v$ for $|v| < 1/2$.

We will write $\widetilde{P}^0$ and $\widetilde{P}^1$ for $\widetilde{P}^{\varphi_0}$ and $\widetilde{P}^{\varphi_1}$, and denote by π^0 and π^1 the respective projections $\widetilde{P}^0 \to \widetilde{P}$ and $\widetilde{P}^1 \to \widetilde{P}$. Similarly, we will use the notation Z^0, Z^1, R^0 and R^1 instead of Z^{φ_0}, Z^{φ_1}, R^{φ_0} and R^{φ_1}. The map $\widetilde{P}^0 \to \widetilde{P}$ is nonsingular, while the map $\widetilde{P}^1 \to \widetilde{P}$ has a fold singularity with image $\partial P \subset \widetilde{P}$. The index of this fold with respect of the outward coorientation to the boundary of P is equal to k.

Taking linear interpolations between φ_0 and φ_1 in one order or the other, we get folded cobordisms in two directions between the map $\widetilde{P}^0 \to \widetilde{P}$ and $\widetilde{P}^1 \to \widetilde{P}$. We will denote the corresponding cobordisms by $\widetilde{P}^{01}$ and $\widetilde{P}^{10}$, respectively. The projections $\pi^{01}\colon \widetilde{P}^{01} \to \widetilde{P}\times[0,1]$ and $\pi^{10}\colon \widetilde{P}^{10} \to \widetilde{P}\times[0,1]$ are folded bordisms in two directions between $\pi^0\colon \widetilde{P}^0 \to \widetilde{P}$ and $\pi^1\colon \widetilde{P}^1 \to \widetilde{P}$. We think of $\widetilde{P}^{\varphi_t}$ as a one-parameter family of (possibly singular) manifolds, interpolating between $\widetilde{P}^0$ and $\widetilde{P}^1$. Using the trivialization (29), these manifolds all have the same boundary, so π^{01} and π^{10} may be used as local models for cobordisms. They allow us to create, or annihilate a fold component, respectively. We describe the fold eliminating surgery more formally in the next section and leave the formal description of the inverse process of fold creating surgery to the reader. In fact fold creating surgery will not be needed for the proof of the main theorem.

In the context of enriched folded maps there are two versions of fold eliminating surgery. One will be referred to as *membrane eliminating*. In this case the membrane will be eliminated together with the fold. The second one will be referred to as *membrane expanding*. In that case, surgery will spread the membrane over $\bar{P}$, the image of P in the target.

For the membrane eliminating case we choose the submanifold

$$V_- = \{(x_2, \dots, x_{d+1}) = 0, x_1 \le 0\} \cap \widetilde{P}^{10} \subset \widetilde{P} \times H \times [0,1]$$

as a template membrane for the folded bordism $\pi^{10}\colon \widetilde{P}^{10} \to \bar{P}$. Next we choose the subbundles K_- and K_+ spanned by the vector fields $\partial/\partial x_2, \dots, \partial/\partial x_k$ and $\partial/\partial x_{k+1}, \dots, \partial/\partial x_{d+1}$, respectively, as a template framing. Note that with this choice we have $\partial V_- = \partial_-(V_-, K)$ and the index of the membrane V_- is equal to $k-1$.

For the membrane expanding surgery we choose as a template membrane the submanifold

$$V_+ = \{(x_1, \dots, x_k, x_{k+2}, \dots, x_{d+1}) = 0, x_{k+1} \ge 0\} \cap \widetilde{P}^{10} \subset \widetilde{P} \times H \times [0,1]$$

with boundary $\Sigma(\pi^{10})$ as the membrane for the folded bordism $\pi^{10}\colon \widetilde{P}^{10} \to \widetilde{P}\times[0,1]$. We choose the subbundles K_- and K_+ spanned by vector fields $\partial/\partial x_1, \dots, \partial/\partial x_k$ and $\partial/\partial x_{k+2}, \dots, \partial/\partial x_{d+1}$, respectively, as a template framing. Note that with this choice we have $\partial V_+ = \partial_+(V_+, K)$ and the index of the membrane V_+ is equal to k.

Note that in the membrane eliminating case the restriction of the membrane V_- to $\widetilde{P}^1$ projects diffeomorphically onto $P \subset \widetilde{P}$, while in the membrane expanding case the restriction of the membrane V_+ to $\widetilde{P}^1$ projects diffeomorphically onto $\widetilde{P} \setminus \operatorname{Int} P \subset \widetilde{P}$.

3.2.2 Surgery (1) *Membrane eliminating surgery*: Let $(f\colon M \to X, \mathfrak{e})$ be an enriched folded map and (V, K) one of its membranes. Suppose that the framed membrane (V, K) is pure and assume first that $\partial_+(V, K) = \varnothing$, and that the index of the membrane is equal to $k - 1 \geq 0$. Note that in this case the boundary fold ∂V has index k with respect to the outward coorientation of $\partial \overline{V}$. Consider the model enriched folded map $\pi^1\colon \widetilde{P}^1 \to \widetilde{P}$ where P is diffeomorphic to V. Fix a diffeomorphism $\psi\colon P \to \overline{V} = f(V) \subset X$. Let U^1 denote a neighborhood of $\partial P \subset \widetilde{P}^1$. According to Lemma 2.9 there exists an extension $\widetilde{\psi}\colon \widetilde{P} \to X$ of the embedding ψ and an embedding $\Psi\colon U^1 \to M$ such that

- the diagram

(30)
$$\begin{array}{ccc} U^1 & \xrightarrow{\Psi} & M \\ \downarrow{\scriptstyle \pi^1} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{\widetilde{\psi}} & X \end{array}$$

  commutes;
- $\Psi^{-1}(V) = V_- \cap U^1$;
- the canonical framing of the membrane $V_- \cap U^1$ is sent by Ψ to the given framing of the membrane V.

The data needed for eliminating the fold ∂V by surgery consists of an extension of Ψ to all of $\widetilde{P}^1$ such that

- the diagram

(31)
$$\begin{array}{ccc} \widetilde{P}^1 & \xrightarrow{\Psi} & M \\ \downarrow{\scriptstyle \pi^1} & & \downarrow{\scriptstyle f} \\ \widetilde{P} & \xrightarrow{\widetilde{\psi}} & X \end{array}$$

  commutes;
- $\Psi^{-1}(V) = V_-^1 := V_- \cap \widetilde{P}^1$;
- the canonical framing of the membrane V_-^1 is sent by Ψ to the given framing of the membrane V.

Construction 3.10 *Fold eliminating surgery (see Figures 11–12) consists of replacing* $\widetilde{P}^1$ *by* $\widetilde{P}_0$ *with the projection* π^0*. More precisely, cut out* $\Psi(\widetilde{P}^1) \times [0,1]$ *from* $M \times [0,1]$ *and glue in* $\widetilde{P}^{10}$ *along the identification (29). This gives an enriched folded map* $W \to X \times [0,1]$ *which is a cobordism starting at* f*, and ending in an enriched folded map where the fold* ∂V, together with its membrane V, *has been removed.*

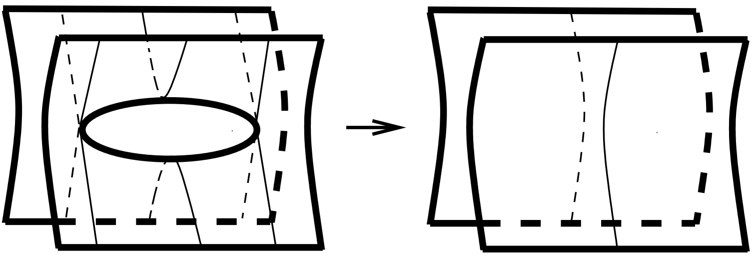


Figure 11: Fold eliminating surgery $(n = 2, d = 0)$

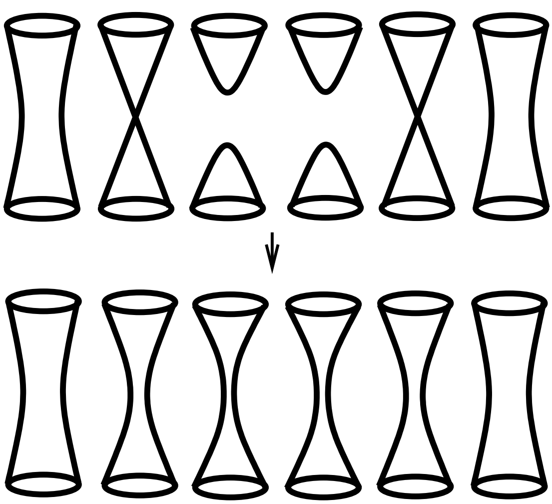


Figure 12: Fold eliminating surgery $(n = 1, d = 2)$

The case $\partial_-(V, K) = \varnothing$ can be reduced to the previous one by the following procedure. Let $\bar{K}$ be the dual framing of V; see Section 2.3 above. Then $\partial V = \partial_-(V, \bar{K})$, $\partial_+(V, \bar{K}) = \varnothing$, $d > k$, and the membrane $(V, \bar{K})$ has index $d - k - 1$. Hence, we can use for the membrane eliminating surgery the above template of index $d - k$, and then switch the framing of the constructed membrane in the cobordism to the dual one.

(2) *Membrane expanding case*: Let P be a domain in X bounded by the image $\bar{C}$ of a fold C of index k with respect to the outward orientation of $\bar{C} = \partial P$. Suppose that the membrane V adjacent to C projects to the complement of P in X, ie $\bar{V} = f(V) \subset X \setminus \operatorname{Int} P$, and $C \subset \partial_+(V, K)$. The case $C \subset \partial_-(V, K)$ can be reduced to the positive by passing to the dual framing as it was explained above in the membrane eliminating case.

Consider the template enriched folded map $\pi^1\colon \widetilde{P}^1 \to \widetilde{P}$ as in Section 3.2.1. Let ψ denote the inclusion $P \hookrightarrow X$. According to Lemma 2.9, there exists an extension $\widetilde{\psi}\colon \widetilde{P} \to X$ of the embedding ψ and an embedding $\Psi\colon U^1 \to M$ such that

- the diagram

(32)
$$\begin{array}{ccc} U^1 & \xrightarrow{\Psi} & M \\ \downarrow{\scriptstyle \pi^1} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{\widetilde{\psi}} & X \end{array}$$

  commutes;
- $\Psi^{-1}(V) = V_+ \cap U^1$;
- the canonical framing of the membrane $V_+ \cap U^1$ is sent by Ψ to the given framing of the membrane V.

In this case the data needed for eliminating the fold ∂V by surgery consists of an extension of Ψ to all of $\widetilde{P}^1$ such that

- the diagram

(33)
$$\begin{array}{ccc} \widetilde{P}^1 & \xrightarrow{\Psi} & M \\ \downarrow{\scriptstyle \pi^1} & & \downarrow{\scriptstyle f} \\ \widetilde{P} & \xrightarrow{\widetilde{\psi}} & X \end{array}$$

  commutes;
- the canonical framing of the membrane $V_+ \cap \widetilde{P}^1$ is sent by Ψ to the given framing of the membrane V.

Construction 3.11 *Fold eliminating surgery consists of replacing $\widetilde{P}^1$ by $\widetilde{P}^0$ with the projection π_0. Exactly as in Construction 3.10 we get an enriched folded map $W \to X \times [0,1]$ which is a cobordism starting at f, and ending in an enriched folded map where the fold ∂V has been removed.*

Both constructions eliminate the fold ∂V. The difference between them is that in the membrane expanding case, the above surgery spreads the membrane V over the domain P.

3.2.3 Bases for fold surgeries The embedding $\Psi\colon \widetilde{P}^1 \to M$ required for the surgeries in Constructions 3.10 and 3.11 is determined up to isotopy by slightly simpler data which we now describe. To any smooth manifold P with collared boundary, let $\varphi_1\colon P \to [-1,1]$ be the function defined in Section 3.2.1 and let

$$S^{k-1}P = \{(p, x_-) \in \widetilde{P} \times D^k \mid \varphi_1(p) = -\|x_-\|^2\}.$$

This is a closed manifold, which up to diffeomorphism depends only on P. In fact, it is diffeomorphic to the boundary of $P \times D^k$ (after smoothing the corners of $P \times D^k$). The projection $(p, x_-) \mapsto p$ restricts to a folded map $\pi\colon S^{k-1}P \to P$ with fold ∂P of index k with respect to the outward orientation of the boundary of P. We have an embedding

$$S^{k-1}P \to \widetilde{P}^1 \subset \widetilde{P} \times H$$

given by $(p, x_-) \mapsto (p, x_-, 0)$. The normal bundle of this embedding has a canonical frame given by projections of the frame

$$\frac{\partial}{\partial x_+} = \left(\frac{\partial}{\partial x_{k+1}}, \dots, \frac{\partial}{\partial x_{d+1}}\right)$$

to $TS^{k-1}P$.

We also have an embedding $\partial P \to S^{k-1}P$ as $p \mapsto (p, 0)$, which identifies ∂P with the folds of the projection $S^{k-1}P \to P$, and the normal bundle of $\partial P \subset S^{k-1}P$ is framed by

$$\frac{\partial}{\partial x_-} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right).$$

If we write $P_- = V_- \cap S^{k-1}P$, we thus have

$$P_- = S^{k-1}P \cap \{(x_2, \dots, x_k) = 0, x_1 \le 0\} = \{(x_2, \dots, x_k) = 0, x_1 = -\sqrt{-\varphi_1(p)}\}.$$

Definition 3.12 Let $(f\colon M^{n+d} \to X^n, \mathfrak{e})$ be an enriched folded map.

(a) *Membrane eliminating case*: Suppose $(V, K) \subset M$ is a framed membrane with $\partial_+(V, K) = \varnothing$. A *basis* for membrane-eliminating surgery consists of a pair $(h\colon S^{k-1}P \to M, \mu)$, where P is a compact n–manifold with boundary, $h\colon S^{k-1}P \to M$ is an embedding, and $\mu = (\mu_{k+1}, \dots, \mu_{d+1})$ is a framing of the normal bundle of h, such that the following conditions are satisfied.

- $h(P_-) = V$.
- The map $f \circ h$ factors through an embedding $g\colon P \to X$, ie $f \circ h = g \circ \pi$, and hence $f(h(S^{k-1}P)) = \bar{V} = f(V)$.
- The vectors $\mu_{k+1}, \dots, \mu_{d+1}$ belong to $\operatorname{Ker} df|_{h(S^{k-1}P)}$ and along $h(P_-)$ they coincide with the given framing of the bundle $\widetilde{\operatorname{Ker}}_+ df$.
- The vectors $dh(\partial/\partial x_2), \dots, dh(\partial/\partial x_k)$ define the prescribed framing of the bundle $\widetilde{\operatorname{Ker}}_-|_V$.T
- $h(S^{k-1}P)$ is disjoint from membranes of $\mathfrak{e}$, other than V.

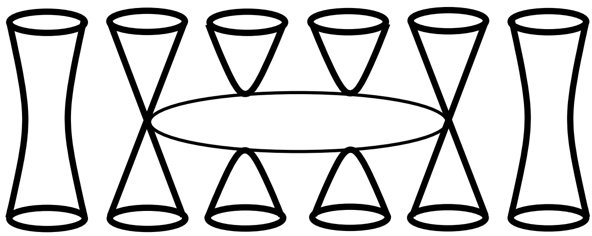


Figure 13: The oval is the image $h(S^{k-1}P)$, where $k = 1$, $P = I$

(b) *Membrane expanding case*: Let P be a domain in X bounded by folds of index k with respect to the outward orientation of $\bar{C} = \partial P$. Let C be the union of the corresponding fold components, and (V, K) the union of framed membranes adjacent to C. Suppose that $C \subset \partial_+(V, K)$ and $f(V) \subset X \setminus \operatorname{Int} P$. A *basis* for membrane-expanding surgery consists of a pair $(h\colon S^{k-1}P \to M, \mu)$, where $h\colon S^{k-1}P \to M$ is an embedding, and $\mu = (\mu_{k+1}, \dots, \mu_{d+1})$ is a framing of the normal bundle of h, such that the following conditions are satisfied.

- The map $f \circ h$ factors through an embedding $g\colon P \hookrightarrow X$, ie $f \circ h = g \circ \pi$.
- $dh(\partial/\partial x_+|_{\partial P}) \subset \operatorname{Ker}_- df$ and coincides with the given framing of the bundle Ker_- over the fold $C = h(\partial P)$.
- $h(S^{k-1}P) \cap V = C$ and $h(S^{k-1}P)$ is disjoint from any other membranes different from V.
- The vector field
$$dh\left((-1)^k \frac{\partial}{\partial x_{k+1}}\right)\Big|_C$$
is tangent to V and inward transversal to ∂V.

Note that in both cases it follows from the above definitions that $f \circ h\colon S^{k-1}P \to X$ is a folded map with the definite fold $\Sigma(f \circ h) = h(\partial P)$.

Given a basis (h,μ), the embedding h can be extended, uniquely up to homotopy, to an embedding $\Psi\colon \widetilde{P}^1 \to M$ such that the diagram (31) or (33) commutes. This, in turn, enables us to perform a membrane eliminating or membrane expanding surgery.

Remark 3.13 (Fold creating surgeries) Fold creating surgeries are inverse to fold eliminating surgeries. For our purposes we will need only one such surgery which creates a fold of index 1 with respect to the *inward* coorientation of its membrane. A basis of such a surgery is given by a pair (h,μ), where h is an embedding $h\colon P\times\{-1,1\}\to M$ over a domain $P\subset X$ disjoint from the folds of the map f, and μ is a framing of the vertical bundle $\operatorname{Ker} df|_{h(P\times\{-1,1\})}$. See Figure 14.

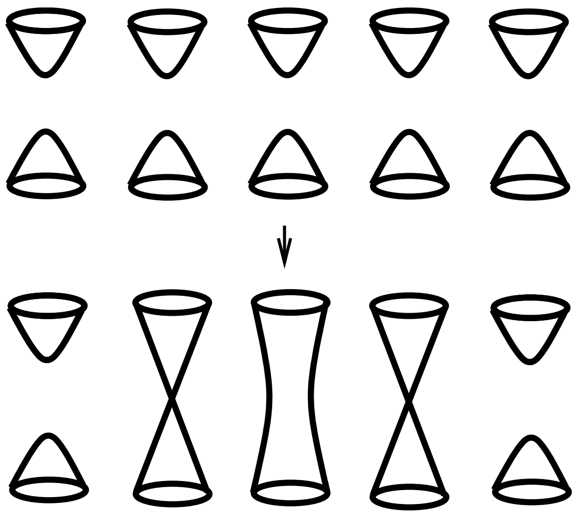


Figure 14: Fold creating surgery

3.2.4 The case $d=2$ Let us review fold eliminating surgeries in the case $d=2$. These surgeries can be of index 0, 1, 2 or 3. Let C be a union of fold components whose projections bound a domain $P\subset X$. In the membrane eliminating case, P is the projection $\overline{V}=f(V)$ of the membrane which spans C. In the membrane expanding case the membranes adjacent to C projects to the complement of the domain P. All fold indices below are with respect to the outward coorientation of the boundary of the domain P.

- *Index 0*: We have $S^{-1}P=\partial P$, ie the basis of the surgery in this case is a framed embedding $h\colon \partial P\to M$ which sends ∂P to the fold C. Only the membrane expanding surgery is possible in this case. When a point $p\in X$ approaches ∂P from outside, a spherical components of the fiber $f^{-1}(p)$ dies. The surgery prevents it from dying end prolongs its existence over all points of P.

- *Index 1*: The surgery basis in this case consists of 2 sections $s_\pm\colon P \to M$, together with framings of the bundle $\operatorname{Ker} df$ over them. As $p \in P$ approaches a point $\overline{z} \in \partial P$, the sections $s_\pm(p)$ converge to the same point $z \in C$, $f(z) = \overline{z}$. In the membrane eliminating case, one of these sections is the membrane V.
  The manifold M' is obtained by a fiberwise index 1 surgery (ie the connected sum) along the framed points $s_+(p)$ and $s_-(p)$, $p \in P$. This eliminates the fold C together with the membrane V in the membrane eliminating case, and spreads the membrane over P in the membrane expanding case. In the latter case the newly created membrane is a section over P which takes values in the circle bundle over P formed by central circles of added cylinders $S^1 \times [-1,1]$.
- *Index 2*: The surgery basis in this case is a circle-subbundle over the domain $P \subset X$, ie a family of circles in fibers $f^{-1}(p)$, $p \in P$, which collapse to points in C when p converges to a boundary point of P. In the fold eliminating case, the membrane V is a section over P of this circle bundle.
  The surgery consists of fiberwise index 2 surgery of fibers along these circles, which eliminates the fold C together with its membrane in the eliminating case, and spreads it over P in the expanding one.
- *Index 3*: The basis of the surgery in this case is a connected component of M which forms an S^2–bundle over P. The 2–spheres collapse to points of C when approaching the boundary of P. The surgery eliminates this whole connected component, in particular removing the fold and its membrane. The membrane expanding case is not possible for $k = 3$.

3.3 Elimination of elliptic folds (proof of Proposition 3.6)

Let $(f, \mathfrak{e})$ be an enriched folded map with $f\colon M \to X$. First, we get rid of nonhyperbolic folds. Let Z be a nonhyperbolic fold component. The following procedure, which is illustrated in Figure 15, replaces Z by a parallel hyperbolic fold.

Let V be the membrane of Z, and $N = \overline{Z} \times [-2,0] \subset X$ be an interior collar of $\overline{Z} = \overline{Z} \times 0$ in $X \setminus \operatorname{Int} \overline{V}$. Let us recall that according to Condition (C1) the map f has a standard end, where it is equivalent to the trivial fibration $T_\infty \times X \to X$. Let $\overline{A} = \overline{Z} \times [-2,-1] \subset N$ and $\overline{B} = \overline{Z} \times [-1,0] \subset N$, so that $N = \overline{A} \cup \overline{B}$. Let us lift $\overline{A}$ to an annulus $A = z \times A \subset M$, $z \in T_\infty$. Write $\overline{Z}_1 = Z \times (-1)$, $\overline{Z}_2 = Z \times (-2)$, $Z_1 = z \times \overline{Z}_1$, $Z_2 = z \times \overline{Z}_2$, so that $\partial \overline{A} = \overline{Z}_1 \cup \overline{Z}_2$ and $\partial A = Z_1 \cup Z_2$. Using Lemma 2.1 we can create a double fold with the membrane A and folds Z_2 of index 1 and Z_1 of index 0 with respect to the coorientation of $\overline{Z}_1$ and $\overline{Z}_2$ by the second coordinate of the splitting $N = Z \times [-2,0]$. The folds Z_1 and Z have index 0 with respect to the outward coorientation of $\partial \overline{B}$, and hence we can use a membrane expanding surgery to kill both folds, Z and Z_1, and spread their membranes over $\bar{B}$. As a result of this procedure we have replaced Z by a hyperbolic fold Z_2.

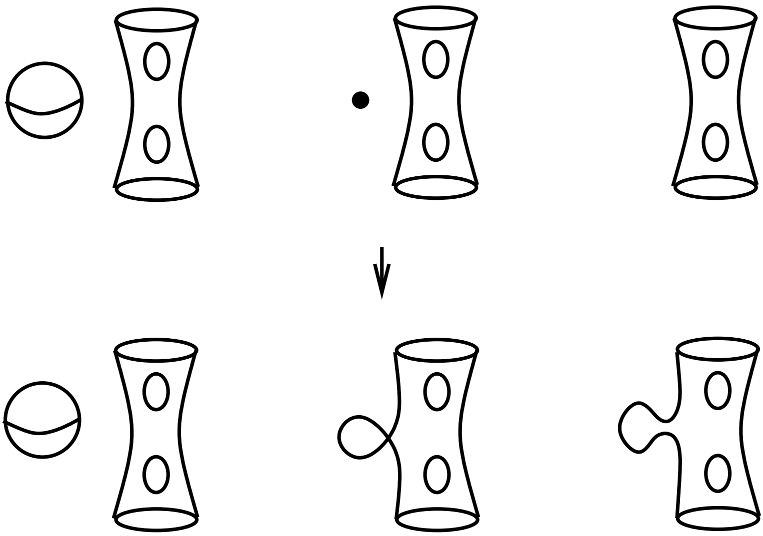


Figure 15: Replacing an elliptic fold by a hyperbolic one

It remains to make the fibers of $f|_{M\setminus\Sigma(f)}$ connected. We begin with the following lemma from [10].

Lemma 3.14 *Let $M \to X$ be an enriched folded map without folds of index 0. Then there exist disjoint n–disks D_i, $i = 1, \dots, K$, embedded into M, such that*

- *$f|_{D_i}$ is embedding $D_i \to \operatorname{Int} X$ for each $i = 1, \dots K$;*
- *$V \cap \bigcup_1^K D_i = \varnothing$, where $V \subset M$ is the union of all membranes;*
- *for each $x \in X$ each irreducible component of $\pi^{-1}(x)$ intersects at least one of the disks D_i at an interior point.*

Proof Note first, that the statement is evident for any *fixed* $x \in X$. Hence, without controlling the disjointness of the disks D_i the statement just follows from the compactness of X. One can choose the required *disjoint* disks D_i using the following trick: fix a function $h\colon M \to \mathbb{R}$ and take the disks D_i such that each disk belongs to its own level hypersurface of h. When we choose such disks for $x \in X$ one needs to avoid the points $z \in F_x = f^{-1}(x)$ where the level hypersurface is tangent to the fiber F_x. It can be done by a small perturbation of disks, if the complement of all "bad" points (for all $x \in X$) is *open* and *dense* in F_x for all $x \in X$. But Thom's jet transversality theorem asserts that this is a generic situation for functions $h\colon M \to \mathbb{R}$. □

Let the disks $D_i \subset M$ be as in Lemma 3.14. Let us also consider disks $\Delta_i = \bar{D}_i \times y_i$, $i = 1, \dots, K$, where $y_1, \dots y_K$ are disjoint points at the end $\bar{T}_\infty$ of the fiber F. Next, using each pair (D_i, Δ_i), $i = 1, \dots, K$, as a basis for an index 1 fold creating surgery we create new folds $\Sigma_i = \partial \widetilde{D}_i$ of index 1 with the disks $\bar{D}_i$ serving as membranes, while making all the fibers connected; see Remark 3.13.

As a result of this step we arrange all fibers $F_x = f^{-1}(x) \setminus \Sigma(f), x \in X$, to be connected. This completes the proof of Proposition 3.6. □

4 Generalized Harer stability theorem. The end of the proof of Theorem 1.8

In this section we show how to deduce Theorem 4.1, the generalized Harer stability theorem, from Theorem 1.9, the ordinary Harer stability theorem. In Section 4.4 below we will use Theorem 4.1 to prove Proposition 3.7 and complete the proof of our main result Theorem 1.8.

4.1 Harer stability for enriched folded maps

The following theorem is the main result of this section. It will be proved in Section 4.3, after some necessary preliminary constructions are introduced in Section 4.2.

Theorem 4.1 *Let $(f\colon M \to X, \mathfrak{e}) \in \mathsf{Fold}_h^\$$ be an enriched folded map. Let $U \subset X$ be a closed domain with smooth boundary transversal to the images of the folds and $\Sigma_1 \subset \Sigma_2$ be compact surfaces with boundary. Let*

$$j\colon (\partial U \times \Sigma_2) \cup (U \times \Sigma_1) \to M$$

be a fiberwise embedding over U whose image does not intersect any fold or membrane and such that the complement of its image in each fiber is connected, even after removing the folds of f.

Then, after possibly changing $(f, \mathfrak{e})$ by a bordism in $\mathsf{Fold}_h^\$$ which is constant outside $\operatorname{Int} U$, the embedding j extends to an embedding of $U \times \Sigma_2$ into M, whose complement in each fiber is connected, even after removing folds.

The following corollary of Theorem 4.1 will be the key ingredient in the proof of Proposition 3.7.

Corollary 4.2 *Let $(f\colon M \to X, \mathfrak{e}) \in \mathsf{Fold}_h^{\$}$ be an enriched folded map. Let $V \subset M$ be one of its membranes and let $C \subset \partial V$ be a union of fold components, all of the same sign. Assume that $\bar{C} = f(C) = \partial P$, for some domain $P \subset X$ and that one of the following conditions hold.*

(M1) *P is a union of components of V.*

(M2) *$P \subset X \setminus \operatorname{Int} V$.*

Then $(f, \mathfrak{e})$ is bordant in the category $\widetilde{\mathsf{Fold}}_h^{\$}$ to an element $(\tilde{f}, \tilde{\mathfrak{e}})$ such that

- *the bordism is constant over the complement of* $\operatorname{Int} P$*;*
- *$(\tilde{f}, \tilde{\mathfrak{e}})$ has no more membranes than $(f, \mathfrak{e})$;*
- *$\tilde{f}$ admits a basis for a surgery eliminating the fold C.*

The surgery eliminates the membrane of C in case (M1) *and spreads it over P in case* (M2).

Proof Consider a slightly smaller domain $U \subset \operatorname{Int} P$, so that $P \setminus \operatorname{Int} U$ is an interior collar of ∂P in P. If the index of C with respect to the outward coorientation of $\bar{C}$ is 1 then the 0–dimensional vanishing cycles over points of ∂U form two sections $s_\pm\colon \partial U \to M$ of the map f. In case (M1) we can assume that one of these sections, say s_-, consists of points of the membrane V. The local structure near the membrane allows us to construct fiberwise embeddings $S_-\colon U \times D^2 \to M$ and $S_+\colon \partial U \times D^2 \to M$ such that $S_-|_{U\times 0}$ extends the section s_-, $S_+|_{\partial U\times 0} = s_+$ and $S_-(U \times 0) \subset V$. Applying Theorem 4.1 with $\Sigma_1 = D^2$, $\Sigma_2 = \Sigma_1 \amalg D^2$, and $j = S_+ \amalg S_-$, we construct a basis for a membrane eliminating surgery which removes the fold C. In the case (M2), the enrichment structure for the membranes adjacent to C provides an extension of the sections $s_\pm$ and s_+ to disjoint fiberwise embeddings $S_\pm\colon \partial U \times D^2 \to M$ such that $S_\pm|_{\partial U' \times 0} = s_\pm$. To conclude the proof in this case we apply Theorem 4.1 with $\Sigma_1 = \varnothing$, $\Sigma_2 = D^2 \amalg D^2$, and $j = S_+ \amalg S_-$. Suppose now that the index of C is 2. Consider first the case (M2). Then the vanishing cycles over ∂U define a fiberwise embedding $\partial U \times S^1 \to M$ over ∂U which extends to a fiberwise embedding $j\colon \partial U \times A \to M$ disjoint from all folds and their membranes, where A is the annulus $S^1 \times [-1, 1]$. It follows from the definition of the category $\mathsf{Fold}_h^{\$}$ that the complement of the image of j is fiberwise connected, even after all singularities are removed. Hence we can apply Theorem 4.1 with $\Sigma_1 = \varnothing$ and $\Sigma_2 = A$ to construct a basis for a membrane expanding surgery eliminating the fold C. Finally, in the case (M1) each vanishing cycle $j(x \times (S^1 \times 0))$, $x \in \partial U$, has a unique point p_x which is also in the membrane V of the fold C. This point is the center of an embedded disk $D^2 \to A$, and the framed membrane gives a fiberwise embedding $j_1\colon U\times D^2\to M$ over U, which over the boundary extends to a fiberwise embedding $j_2\colon \partial U\times A\to M$ over ∂U. Hence, we are in a position to apply Theorem 4.1 with $\Sigma_1=D^2$ and $\Sigma_2=A$. □

4.2 Nodal surfaces and their unfolding

Let $Q\colon \mathbb{R}^3\to\mathbb{R}$ be the quadratic form

(34) $$Q(x)=x_1^2+x_2^2-x_3^2,$$

let $H=\{x\in\mathbb{R}^3 \,|\, |Q|\le 1; |x_3|\le 2\}$ and denote by K_t the level set $\{Q=t\}\cap H$, $t\in[-1,1]$. When passing through the critical value 0, the level set K_t experiences a surgery of index 1, ie changes from a 2–sheeted to a 1–sheeted hyperboloid. The critical level set K_0 is the cone $\{x_1^2+x_2^2-x_3^2=0,\ |x_3|\le 2\}$. Let us fix diffeomorphisms $\beta_\pm\colon \partial_\pm H=H\cap\{x_3=\pm 2\}\to S^1\times[-1,1]$ which send the boundary circles of K_t to $S^1\times\{t\}$, $t\in I=[-1,1]$. We will write $\beta_\pm(x)=(\beta_\pm^S(x),\beta_\pm^I(x))\in S^1\times[-1,1]$ for $x\in\partial_\pm H$.

We will call the singularity of K_0 a *node* and call surfaces with such singularities *nodal*. A singular surface S is called k*–nodal* if it is a smooth surface in the complement of k points $p_1,\dots,p_k\in S$, while each of these points has a neighborhood diffeomorphic to K_0.

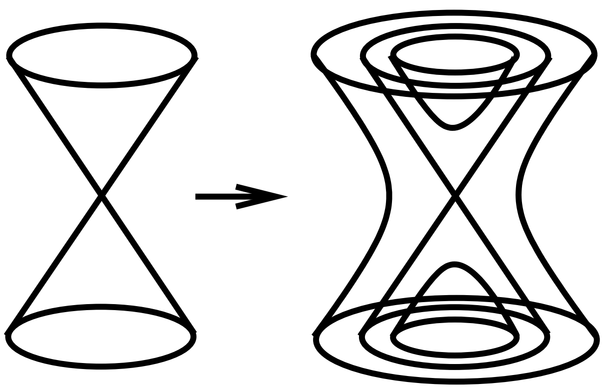


Figure 16: Nodal surface and its unfolding

The function Q is a folded map $H\to\mathbb{R}$ with a single fold point at the origin $0\in H$. If we want to promote it to an *enriched* folded map, there are four local possibilities for the choice of an enriched membrane depending on the membrane index and the sign of Σ as the membrane boundary component:

(i) *Index* 0 *membrane with negative boundary*: $V_0^- = \{x_1, x_2 = 0, x_3 \le 0\}$, $K_- = \{0\}$, $K_+ = \mathrm{Span}(\partial/\partial x_1, \partial/\partial x_2)$;

(ii) *Index* 1 *membrane with positive boundary*: $V_1^+ = \{x_2, x_3 = 0, x_1 \ge 0\}$, $K_- = \mathrm{Span}(\partial/\partial x_3)$, $K_+ = \mathrm{Span}(\partial/\partial x_2)$;

(iii) *Index* 1 *membrane with negative boundary*: $V_1^- = \{x_2, x_3 = 0, x_1 \le 0\}$, $K_- = \mathrm{Span}(\partial/\partial x_2)$, $K_+ = \mathrm{Span}(\partial/\partial x_3)$

(iv) *Index* 2 *membrane with positive boundary*: $V_2^+ = \{x_1, x_2 = 0, x_3 \ge 0\}$, $K_+ = \{0\}$, $K_- = \mathrm{Span}(\partial/\partial x_1, \partial/\partial x_2)$.

In the cases (i)–(ii), the fold has index 1 and in the cases (iii)–(iv) it has index 2, with respect to the outward orientation of the boundary of the projection $\overline{V}$ of V.

A k*–nodal fibration* $f\colon Y \to Z$ is a fiber bundle whose fibers are k–nodal surfaces, equipped with k disjoint fiberwise embeddings $\psi_i\colon Z \times K_0 \to Y$ over Y, such that the complement of the images of the ψ_i forms a smooth fiber bundle over Z.

Let $f\colon Y \to Z$ be a k–nodal fibration and write $\widehat{Z} = Z \times I^k$. We construct a manifold $\widehat{Y}$ together with a map $\widehat{f}\colon \widehat{Y} \to \widehat{Z}$ as follows. Set

$$\widehat{Y} = \left(Y \setminus \bigcup_1^k \psi_i(Z \times K_0)\right) \times I^k \underset{\sigma_1}{\cup} (Z \times H \times I^{k-1}) \underset{\sigma_2}{\cup} \cdots \underset{\sigma_k}{\cup} (Z \times H \times I^{k-1}),$$

where $\sigma_i\colon Z \times (\partial_+ H \cup \partial_- H) \times I^{k-1} \to \psi_i(Z \times \partial K_0) \times I^k$, $i = 1, \dots, k$, are gluing diffeomorphisms defined by the formula

$$\sigma_i(z, x, t_1, \dots, t_{k-1}) = \psi_i(z, \beta^S_\pm(x), t_1, \dots, t_{i-1}, \beta^I_\pm(x), t_i, \dots, t_{k-1})$$

for $x \in \partial_\pm H$, $z \in Z$ and $t_j \in I$ for $j = 1, \dots, k-1$. The map $f\colon Y \to Z$ extends to a map $\widehat{f}\colon \widehat{Y} \to \widehat{Z}$ which is equal to the projection $(y, t) \mapsto (f(y), t) \in \widehat{Z} = Z \times I^k$ for $(y, t) \in \left(Y \setminus \bigcup_1^k \psi_i(K_0 \times Z)\right) \times I^k$ and equal to the map

$$(z, x, t_1, \dots, t_{k-1}) \mapsto (z, t_1, \dots, t_{i-1}, Q(x), t_i, \dots, t_k)$$

on the i–th copy of $Z \times H \times I^{k-1}$ glued with the attaching map σ_i. Note that the map $\widehat{f}$ has k fold components which are mapped to the hypersurfaces $C_i = \{t_i = 0\} \subset \widehat{Z} = Z \times I^k$. Thus $Z = Z \times 0 = \bigcap_1^k \overline{C}_i$ is the locus of k–fold intersection of images of fold components of $\widehat{f}$. We will call the folded map $\widehat{f}\colon \widehat{Y} \to \widehat{Z}$ the *universal unfolding* of the k–nodal fibration $f\colon Y \to Z$; see Figure 16.

The following lemma, which follows from the local description of an enriched folded map in a neighborhood of its fold; see Lemma 2.9, shows that the universal unfolding describes the structure of a folded map over a neighborhood of the locus of maximal multiplicity of fold intersection.

Lemma 4.3 *Let $f\colon M \to X$ be a folded map with cooriented hyperbolic folds. Suppose that all combinations of (projections of) fold components intersect transversally among themselves and with ∂X. Let k be the maximal multiplicity of the fold intersection and let Z be one of the components of the k–fold intersection. Then $Z \subset X$ is a submanifold with boundary $\partial Z \subset \partial X$, and the restriction $f|_{Y=f^{-1}(Z)}\colon Y \to Z$ is a k–nodal fibration. Let $\widehat{f}\colon \widehat{Y} \to \widehat{Z} = Z \times I^k$ be the universal unfolding of the k–nodal fibration $f|_Y$. Then there exist embeddings $\varphi\colon \widehat{Z} \to X$ and $\Phi\colon \widehat{Y} \to M$ which extend the inclusions $Z \hookrightarrow X$ and $Y \hookrightarrow M$ such that the diagram*

$$\begin{array}{ccc} \widehat{Y} & \xrightarrow{\Phi} & M \\ \downarrow{\scriptstyle \widehat{f}} & & \downarrow{\scriptstyle f} \\ \widehat{Z} & \xrightarrow{\varphi} & X \end{array} \tag{35}$$

commutes. If the folded map is enriched then one can arrange that the preimages of the membranes and their framings under the embedding $\Phi\colon \widehat{Y} \to M$ coincide with the submanifolds $Z \times V_j^{\pm} \times I^{k-1}$, and their model framings defined above (where $j \in \{0,1,2\}$ and the sign $\pm$ depends on the index of the membrane and the sign of the folds), in the corresponding copies of $Z \times H \times I^{k-1}$ in $\widehat{Y}$.

Harer's stability theorem, in the form of Theorem 1.9 implies the following statement for k–nodal fibrations.

Theorem 4.4 (Geometric form of Harer stability for nodal fibrations) *Let $\Sigma_1 \subset \Sigma_2$ be compact surfaces with boundary (not necessarily connected). Let $f\colon M \to X$ be a k–nodal fibration with T_∞ ends, and let*

$$j\colon (\partial X \times \Sigma_2) \cup (X \times \Sigma_1) \to M$$

be a fiberwise embedding over X, such that its image in each fiber is disjoint from the nodes, and that in each fiber the complement of its image is connected, even after removing all nodes.

Then, after possibly changing $f\colon M \to X$ by a bordism which is the trivial bordism over ∂X, the embedding j extends to an embedding of $X \times \Sigma_2$, still disjoint from nodes and with connected complement.

4.3 Proof of Theorem 4.1

Let $(f\colon M \to X, \mathfrak{e})$ be an enriched folded map from $\mathsf{Fold}_h^{\$}$. Let $U \subset X$ be a compact domain with smooth boundary. We assume that all combinations of (projections of) fold components intersect transversally among themselves and with ∂U. Let k be the maximal multiplicity of the fold intersection. We denote by U_j, $j = 1, \dots, k$, the set of intersection points of multiplicity $\geq j$ in U, and set $U_0 = U$. Thus we get a stratification $U = \bigcup_0^k U_j \setminus U_{j+1}$. Set $M_j = f^{-1}(U_j)$. Note that U_k is a closed submanifold of U with boundary $\partial U_k \subset \partial U$. The map $f_k = f|_{M_k}\colon M_k \to U_k$ is a k–nodal fibration. The membranes which are not adjacent to the fold components intersecting along U_k, together with their framings define fiberwise embeddings $s_1, \dots, s_l\colon U_k \times D^2 \to M_k$ over U_k, disjoint from the image of j and from each other.

Let us apply Theorem 4.4, the nodal version of Harer's stability theorem, to the nodal fibration f_k and the fiberwise embedding

$$\tilde{j} = j \sqcup \bigcup_1^l s_i\colon (\partial U_k \times \tilde{\Sigma}_2) \cup (U_k \times \tilde{\Sigma}_1) \to M_k,$$

where $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are disjoint unions of Σ_1 and Σ_2, respectively, with l copies of the disk D^2. As a result, we find a bordism $F_k\colon W_k \to Y_k$ (in the class of k–nodal fibrations) between the k–nodal fibrations $f_k\colon M_k \to U_k$ and $f_k'\colon M_k' \to U_k'$. Precisely, we have a partition $\partial Y_k = \partial_- Y_k \cup \partial_+ Y_k$, where $\partial_- Y_k = U_k$, $\partial_+ Y_k = U_k'$, and $\partial_- Y_k \cap \partial_+ Y_k = \partial U_k = \partial U_k'$, and the fiberwise embedding $\tilde{j}$ extends to a fiberwise embedding over U_k

$$J\colon (\partial_+ Y_k \times \tilde{\Sigma}_2) \cup (Y_k \cup \tilde{\Sigma}_1) \to W_k.$$

In addition, we can arrange that $\pi_0(U_k) \to \pi_0(Y_k)$ is injective (for example, this could be arranged by applying Theorem 4.4 to each path component of U_k separately).

Let $\widehat{F}_k\colon \widehat{W}_k \to \widehat{Y}_k = Y_k \times I^k$ be the universal unfolding of the k–nodal fibration $F_k\colon W_k \to Y_k$. We view $\widehat{F}_k$ as a bordism between the universal unfoldings $\hat{f}_k\colon \widehat{M}_k \to \widehat{U}_k = U_k \times I^k$ and $\hat{f}_k'\colon \widehat{M}_k \to \widehat{U}_k' = U_k' \times I^k$ of the k–nodal fibrations f_k and f_k'. The embedding J extends to a fiberwise embedding

$$\widehat{J}\colon (\partial_+ \widehat{Y}_k \times \tilde{\Sigma}_2) \cup (\widehat{Y}_k \times \tilde{\Sigma}_1) \to \widehat{W}_k$$

over $\widehat{Y}_k$. According to Lemma 4.3 the restriction of f to a tubular neighborhood of U_k is isomorphic to the universal unfolding $\hat{f}_k\colon \widehat{M}_k \to \widehat{U}_k$ of the k–nodal fibration f_k. In other words, there exist embeddings $\varphi_k\colon \widehat{U}_k \to X$ and $\Phi_k\colon \widehat{M}_k \to M$ which extend the inclusions $U_k \hookrightarrow X$ and $M_k \hookrightarrow M$ such that the diagram

$$\begin{array}{ccc} \widehat{M}_k & \xrightarrow{\Phi_k} & M \\ \downarrow{\scriptstyle \hat{f}_k} & & \downarrow{\scriptstyle f} \\ \widehat{X}_k & \xrightarrow{\varphi_k} & X \end{array} \tag{36}$$

commutes. Moreover, (φ_k, Φ_k) can be chosen in such a way that the framed membranes adjacent to intersecting folds correspond to model framed membranes of Lemma 4.3.

Let us glue the bordism $\widehat{F}_k\colon \widehat{W}_k \to \widehat{Y}_k$ to the trivial bordism $F = f \times \mathrm{Id}\colon W = M \times I \to Y = X \times I$ using the attaching maps (φ_k, Φ_k). We get a cobordism

$$\widetilde{F}\colon \widehat{W}_k \underset{\Phi_k \times 1}{\cup} M \times I \to \widehat{Y}_k \underset{\varphi_k \times 1}{\cup} X \times I.$$

Strictly speaking, the gluing produces a manifold with corners, but we smooth these in the usual way. The folded map $\widetilde{F}\colon \widetilde{W} \to \widetilde{Y}$ resulting from this construction is a bordism between $f\colon M \to X$ and $f'\colon M' \to X'$, which is trivial over the complement of $\varphi_k(\widehat{U}_k) \subset X$. The model framed membranes of Lemma 4.3 give us a canonical extension of the framed membranes adjacent to Y_k to all of $\widehat{W}_k$. On the other hand, the restriction of $\widehat{J}$ to $\widehat{Y}_k \times \coprod^l D^2 \subset \widehat{Y}_k \times \Sigma_2$ allows us to extend all the other framed membranes. It remains to see that the new membranes $V \subset M'$ have $\partial_+ V$ null homologous in X, but this follows from the assumption that $\pi_0(U_k) \to \pi_0(Y_k)$ is injective. Thus we have constructed a map $\widetilde{F}\colon \widetilde{W} \to \widetilde{Y}$ which together with the enrichment is a bordism in the category $\mathsf{Fold}_h^{\$}$.

The fiberwise embedding $\widehat{J}\colon U'_k \times \Sigma_2 \to M'_k$ extends to a closed neighborhood $\Omega \supset U'_k$ in M'. In the domain $U'' = U' \setminus \operatorname{Int} \Omega$, the maximal multiplicity of fold intersection is now $k-1$. Hence, we can repeat the previous argument to extend $\widehat{J}$ over the stratum U''_{k-1}, possibly after changing it by another bordism in the category $\mathsf{Fold}_h^{\$}$. Continuing inductively we find the required extension to the whole domain bounded by ∂U. □

Now we are ready to prove Proposition 3.7. Together with Proposition 3.6, this will complete our proof of Theorem 1.8.

4.4 Elimination of hyperbolic folds (proof of Proposition 3.7)

Let $V_1, \ldots, V_N$ be the collection of framed membranes of $\mathfrak{f} = (f, \mathfrak{e})$. We are going to inductively remove all of them. If the membrane V_1 is pure then, in view of Corollary 4.2, we can assume, after a possible change of f by a bordism in the category $\mathsf{Fold}_h^{\$}$, that there is a basis for the fold surgery which removes ∂V_1 together with the membrane.

Suppose now that the membrane V_1 is mixed. Consider first the case $n > 1$

Let's write $S^1 = \mathbb{R}/4\mathbb{Z}$. A membrane V in X gives rise to a smooth function $\gamma\colon (X, \partial X) \to (S^1, *)$, with the property that $V = \gamma^{-1}([1,3])$ and that γ is transverse to $\{1,3\}$ and that $\partial_\pm V = \gamma^{-1}(2 \pm 1)$. These properties determine the map γ uniquely up to homotopy, and therefore gives a class in $H_{n-1}(X) = H^1(X, \partial X) = [(X, \partial X), (S^1, *)]$. This class is just the image of the fundamental class of $\partial_+ V$ (or equivalently of $\partial_- V$), so γ is null-homotopic. Therefore we get a lift of γ to a map $g\colon (X, \partial X) \to (\mathbb{R}, 0)$ into the universal cover of S^1. This map is transverse to the odd numbers, and ∂V is the union of the submanifolds $C_k = g^{-1}(1+2k)$. The sign of the fold $C \subset \partial V$ is determined by the parity of k. Pick $k \in \mathbb{Z}$ such that $C_k \neq \varnothing$ but that $C_{k'} = \varnothing$ whenever $|k'| > |k|$. If $1+2k$ is positive, C_k is then the boundary of the domain $P = \{g \geq 1+2k\}$ and if $1+2k$ is negative, C_k bounds the domain $P = \{g \leq 1+2k\}$. In either case we can use Corollary 4.2 to create a basis for a surgery eliminating this part of the fold. (The parity of k and sign of $2k+1$ determines whether the surgery eliminates the fold or spreads it over P.) After applying this process finitely many times, the membrane becomes pure and then eliminated. This completes the proof of Proposition 3.7 when $n > 1$.

Finally consider the case $n = 1$, ie $X = I$ or $X = S^1$. We assume for determinacy that $X = I = [0,3]$, ie f is a Morse function. A mixed framed membrane of f connects two critical points p_1, p_2 of f of index 1 and 2, and with critical values c_1, C_2, $c_1 < c_2$, respectively. We may assume $c_1 = 1$ and $c_2 = 2$ and that $\overline{V} = [1,2]$. For a small $\epsilon > 0$ let us consider vanishing circles $S_1 \subset F_{1+\epsilon}, S_2 \subset F_{2-\epsilon}$ of critical points p_1 and p_2, where F_t denotes the fiber $f^{-1}(t)$, $t \in \mathbb{R}$. The embedding of $S_2 \to F_{2-\epsilon}$ extends to an embedding

$$j\colon \Sigma_2 = S^1 \times [-1,1] \to F_{2-\epsilon}$$

which maps $S^1 \times \{0\}$ to S_2. Suppose first that it is possible to extend j to a fiberwise embedding

$$\tilde{j}\colon [1+\epsilon, 2-\epsilon] \times \Sigma_2 \to M$$

over $[1+\epsilon, 2-\epsilon]$, such that the circle $\tilde{j}\colon \{1+\epsilon\} \times S^1 \times \{0\} \to F_{1+\epsilon}$ intersects S_1 transversally in exactly one point, we could apply the standard Morse theory cancellation lemma [11] to kill both critical points.

If such an extension is not immediately possible, we can use Theorem 4.1 to change $f\colon M \to I$ by a bordism, after which it will hold. More precisely, we apply the theorem with $U = [1+\epsilon, 2-\epsilon]$, $\Sigma_1 = D^2$ and $\Sigma_2 = S^1 \times [-1,1]$. Then we have a fiberwise embedding

$$(\partial U \times \Sigma_2) \cup (U \times \Sigma_1) \to M,$$

which on $U \times \Sigma_1 = U \times D^2$ is given by the membrane and its framing, on $\{2-\epsilon\} \times \Sigma_2$ is given by the map j above, and on $\{1+\epsilon\} \times \Sigma_2$ it maps the circle $\{1+\epsilon\} \times S^1 \times \{0\}$ to some circle in $F_{1+\epsilon}$ which intersects the vanishing circle S_1 transversally in exactly one point. Applying Theorem 4.1 to this situation brings us back to the previous situation. □

This completes the proof of Theorem 1.8.

5 Miscellaneous

5.1 Appendix A: From wrinkles to double folds

5.1.1 Cusps Let $n > 1$. Given a map $f\colon M \to X$, a point $p \in \Sigma(f)$ is called a *cusp* type singularity or a *cusp* of index $s + 1/2$ if near the point p the map f is equivalent to the map

$$\mathbb{R}^{n-1} \times \mathbb{R}^1 \times \mathbb{R}^d \to \mathbb{R}^{n-1} \times \mathbb{R}^1$$

given by the formula

$$(y, z, x) \mapsto \left(y, z^3 + 3y_1 z - \sum_1^s x_i^2 + \sum_{s+1}^d x_j^2 \right) \tag{37}$$

where $x = (x_1, \dots, x_d) \in \mathbb{R}^d,\ z \in \mathbb{R}^1,\ y = (y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1}$.

The set of cusp points is denoted by $\Sigma^{11}(f)$. It is a codimension 1 submanifold of $\Sigma(f)$ which is in the above canonical coordinates is given by $x = (x_1, \dots, x_d) = 0$, $y_1 = z = 0$. The vector field $\partial/\partial y_1$ along $\Sigma^{11}(f)$ is called the *characteristic vector* field of the cusp locus. It can be invariantly defined as follows. Note that for any point $p \in \Sigma^{11}(f)$ there exists a neighborhood $\Omega \ni f(p)$ in X such that $\Omega \cap f(\Sigma(f))$ can be presented as a union of two manifolds $\overline{\Sigma}_\pm$ with the common boundary $\partial\overline{\Sigma}_\pm = \Omega \cap f(\Sigma^{11}(f))$, the common tangent space $T = T_{f(p)}\overline{\Sigma}_\pm = df(T_p M)$ at the point $f(p)$, and the common outward coorientation ν of $T' = T_p \partial\overline{\Sigma}_\pm \subset T$. On the other hand, the differential df defines an isomorphism

$$T_p M/(\operatorname{Ker} d_p f + T_p \Sigma(f)) \to T/T'.$$

Hence, there exists a vector field Y transversal to $\operatorname{Ker} df + T\Sigma(f)$ in TM along $\Sigma^{11}(f)$, whose projection defines the coorientation ν of T' in T for all points $p \in \Sigma^{11}(f)$. One can show that any vector field Y defined that way coincides with the vector field $\partial/\partial y_1$ for some local coordinate system in which the map f has the canonical form (37).

Note that the line bundle $\lambda = \operatorname{Ker} df \cap T\Sigma(f)$ over $\Sigma^{11}(f)$ is always trivial. Indeed, λ can be equivalently defined as the kernel of the quadratic form $d^2 f\colon \operatorname{Ker} df \to \operatorname{Coker} df$, and thus one has an invariantly defined cubic form $d^3 f\colon \lambda \to \operatorname{Coker} df$ which does not vanish. The bundle $\operatorname{Ker} df|_{\Sigma^{11}(f)}$ can be split as $\operatorname{Ker}_+ \oplus \operatorname{Ker}_- \oplus \lambda$, so that the quadratic form $d^2 f$ is positive definite on Ker_+ and negative definite on Ker_-.

5.1.2 Wrinkles and wrinkled mappings Consider the map

$$w(n+d,n,s)\colon \mathbb{R}^{n-1}\times\mathbb{R}^1\times\mathbb{R}^d\to\mathbb{R}^{n-1}\times\mathbb{R}^1$$

given by the formula

$$(y,z,x)\mapsto\left(y,z^3+3(|y|^2-1)z-\sum_1^s x_i^2+\sum_{s+1}^d x_j^2\right),$$

where $y\in\mathbb{R}^{n-1},\ z\in\mathbb{R}^1,\ x\in\mathbb{R}^d$ and $|y|^2=\sum_1^{n-1}y_i^2$.

The singularity $\Sigma(w(n+d,n,s))$ is the $(n-1)$–dimensional sphere

$$S^{n-1}=S^{n-1}\times 0\subset\mathbb{R}^n\times\mathbb{R}^d$$

whose equator $\Sigma^{11}(f)=\{|y|=1,z=0,x=0\}\subset\Sigma(w(n+d,n,s))$ consists of cusp points of index $s+1/2$. The upper hemisphere $\Sigma(w)\cap\{z>0\}$ consists of folds of index s, while the lower one $\Sigma(w)\cap\{z<0\}$ consists of folds of index $s+1$. The radial vector field $Y=\sum_1^{n-1}y_j(\partial/\partial y_j)$ serves as a characteristic vector field of the cusp locus.

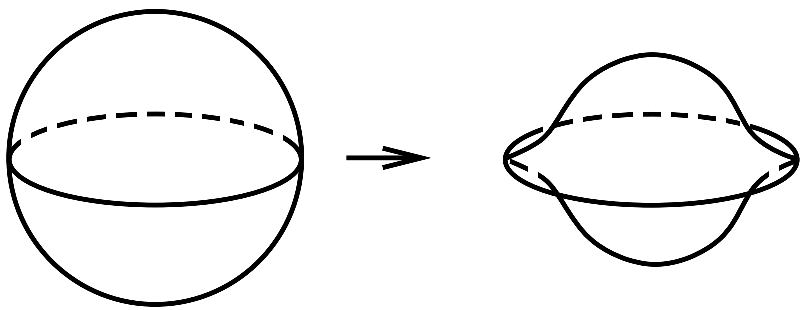


Figure 17: Wrinkle in the source and in the image

Although the differential $dw(n+d,n,s)\colon T(\mathbb{R}^{n+d})\to T(\mathbb{R}^n)$ degenerates at points of $\Sigma(w)$, it can be canonically *regularized* over $\mathcal{O}p_{\mathbb{R}^{n+d}}D^n$, an open neighborhood of the disk $D^n=D^n\times 0\subset\mathbb{R}^n\times\mathbb{R}^d$. Namely, we can substitute for the element $3(z^2+|y|^2-1)$ in the Jacobi matrix of $w(n+d,n,s)$ a function γ which coincides with $3(z^2+|y|^2-1)$ on $\mathbb{R}^{n+d}\setminus\mathcal{O}p_{\mathbb{R}^{n+d}}D^n$ and does not vanish along the n–dimensional subspace $\{x=0\}=\mathbb{R}^n\times\mathbf{0}\subset\mathbb{R}^{n+d}$. The new bundle map $\mathcal{R}(dw)\colon T(\mathbb{R}^{n+d})\to T(\mathbb{R}^n)$ provides a homotopically canonical extension of the map $dw\colon T(\mathbb{R}^{n+d}\setminus\mathcal{O}p_{\mathbb{R}^{n+d}}D^n)\to T(\mathbb{R}^n)$ to an epimorphism (fiberwise surjective bundle map) $T(\mathbb{R}^{n+d})\to T(\mathbb{R}^n)$. We call $\mathcal{R}(dw)$ the *regularized differential* of the map $w(n+d,n,s)$.

A map $f\colon U \to X$ defined on an open ball $U \subset M$ is called a *wrinkle* of index $s+1/2$ if it is equivalent to the restriction $w(n+d,n,s)|_{\mathcal{O}p_{\mathbb{R}^{n+d}} D^n}$. We will use the term "wrinkle" also for the singularity $\Sigma(f)$ of a wrinkle f.

Notice that for $n=1$ the wrinkle is a function with two nondegenerate critical points of indices s and $s+1$ given in a neighborhood of a gradient trajectory which connects the two points. Thus in this case a wrinkle is the same as a double fold.

A map $f\colon M \to X$ is called *wrinkled* if there exist disjoint open subsets $U_1,\dots,U_l \subset M$ such that the restriction $f|_{M\setminus U}$, $U=\bigcup_1^l U_i$, is a submersion (ie has rank equal n) and for each $i=1,\dots,l$ the restriction $f|_{U_i}$ is a wrinkle.

The singular locus $\Sigma(f)$ of a wrinkled map f is a union of $(n-1)$–dimensional spheres (wrinkles) $S_i=\Sigma(f|_{U_i})\subset U_i$. Each S_i has a $(n-2)$–dimensional equator $S_i'\subset S_i$ of cusps which divides S_i into two hemispheres of folds of two neighboring indices. The differential $df\colon T(M)\to T(X)$ can be regularized to obtain an epimorphism $\mathcal{R}(df)\colon T(M)\to T(X)$. To get $\mathcal{R}(df)$ we regularize $df|_{U_i}$ for each wrinkle $f|_{U_i}$.

The following theorem is the main result of the paper [4]:

Theorem 5.1 (Wrinkled mappings) *Let $F\colon T(M)\to T(X)$ be an epimorphism which covers a map $f\colon M\to X$. Suppose that f is a submersion on a neighborhood of a closed subset $K\subset M$, and F coincides with df over that neighborhood. Then there exists a wrinkled map $g\colon M\to X$ which coincides with f near K and such that $\mathcal{R}(dg)$ and F are homotopic rel. $T(M)|_K$. Moreover, the map g can be chosen arbitrarily C^0–close to f and with wrinkles contained in an arbitrarily small balls.*

5.1.3 Cusp eliminating surgery We are going to modify each wrinkle to a spherical double fold using *cusp elimination surgery*, which is one of the surgery operations studied in [3]. Unlike fold elimination surgeries described above in Section 3.2 cusp elimination surgery does not affect the underlying manifold and changes a map by a homotopic one. For maps $\mathbb{R}^2\to\mathbb{R}^2$ the operation is shown on Figure 18.

Definition 5.2 Let $C\subset\Sigma(f)$ be a connected component of the cusp locus. Let Y be the characteristic vector field of C. Suppose that the bundles Ker_-, Ker_+ and λ over C are trivialized, respectively, by the frames

$$\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_s}\right),\quad \left(\frac{\partial}{\partial x_{s+1}},\dots,\frac{\partial}{\partial x_d}\right)\quad\text{and}\quad \frac{\partial}{\partial z}.$$

A *basis* for a cusp eliminating surgery consists of an $(n-1)$–dimensional submanifold $A\subset M$ bounded by C, together with an extension of the above framing as a trivialization of the normal bundle ν of A in M, such that

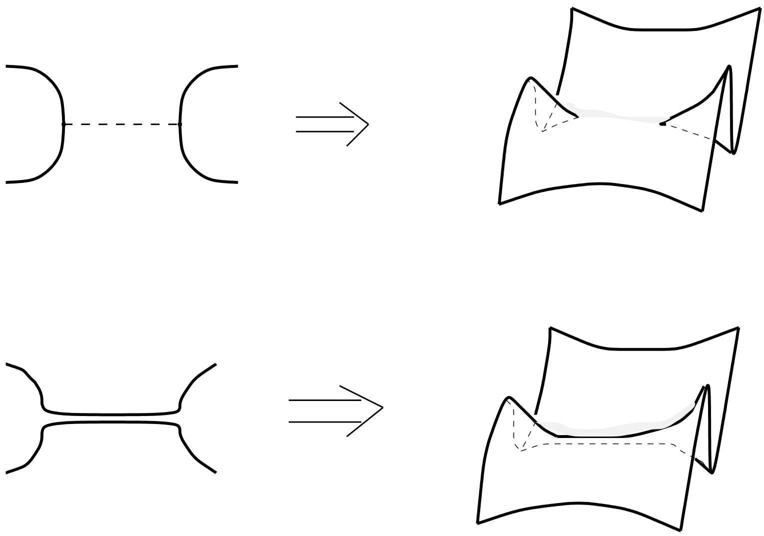


Figure 18: Cusp eliminating surgery in the case $n = 2, d = 0$

- $f|_{\mathrm{Int}\,A}\colon \mathrm{Int}\,A \to X$ is an immersion;
- the characteristic vector field Y is tangent to A along C, and inward transversal to $C = \partial A$;
- $\partial/\partial x_j \in \mathrm{Ker}\,df$ for all $j = 1, \dots, d$.

Let us extend A to a slightly bigger manifold $\widetilde{A}$ ($\dim \widetilde{A} = \dim A$) such that $\mathrm{Int}\,\widetilde{A} \supset A$, and extend the framing over $\widetilde{A}$. One can show (see [3; 1]) that there exists a splitting $U \to \widetilde{A} \times \mathbb{R} \times \mathbb{R}^d$ of a tubular neighborhood of $\widetilde{A}$ in M, such that in the corresponding local coordinates $y \in \widetilde{A}$, $z \in \mathbb{R}$ and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ the map f can be presented as a composition

$$U \xrightarrow{F} \widetilde{A} \times \mathbb{R} \xrightarrow{h} X,$$

where h is an immersion and F has the form

$$F(y, z, x) = \left(y, z^3 + 3\varphi_0(y)\sigma\left(\frac{1}{\epsilon}\left(z^2 + \sum_1^d x_j^2\right)\right)z - \sum_1^s x_i^2 + \sum_{s+1}^d x_j^2 \right),$$

where the function $\varphi_0\colon \mathbb{R} \to \mathbb{R}$ satisfies $\varphi_0 > 0$ on $\mathrm{Int}\,A \subset \widetilde{A}$ and $\varphi_0 < 0$ on $\widetilde{A} \setminus A$, $\sigma\colon [0, 1] \to [0, 1]$ is a cut-off function equal to 1 near 0 and to 0 near 1, and $\epsilon > 0$ is small enough.

Consider another function $\varphi_1\colon \widetilde{A} \to (-\infty, 0)$ which coincides with φ_0 outside $\mathcal{O}p\,A \subset \widetilde{A}$ and such that $|\varphi_1| \le |\varphi_0|$. Let $\varphi_t = (1-t)\varphi_0 + t\varphi_1$, $t \in [0, 1]$, and consider homotopies

$$F_t(y,z,x)=\left(y, z^3+3\varphi_t(y)\sigma\left(\frac{1}{\epsilon}\left(z^2+\sum_1^d x_j^2\right)\right)z-\sum_1^s x_i^2+\sum_{s+1}^d x_j^2\right),$$

$(y,z,x)\in U$, and $f_t=h\circ F_t\colon U\to X$. The homotopy f_t is supported in U and hence can be extended to the whole manifold M as equal to f on $M\setminus U$. The next proposition is straightforward.

Proposition 5.3 (1) *The homotopy f_t removes the cusp component C. The map f_1 coincides with f_0 outside U, has only fold type singularities in U, and*

$$\Sigma(f_1|_U)=\{x=0,\ z^2=-\varphi_1(y)\}.$$

(2) *Suppose that $\Sigma(f)\setminus C$ consists of only fold points and that the restriction of the map f to $\Sigma\cup A$ is an embedding. Then the restriction $f_1|_{\Sigma(f_1)}\colon \Sigma(f_1)\to X$ is an embedding provided that the neighborhood $U\supset A$ in the surgery construction is chosen small enough.*

5.1.4 From wrinkles to double folds

Proposition 5.4 *Let*

$$w(n+d,n,s)\colon \mathbb{R}^{n-1}\times\mathbb{R}^1\times\mathbb{R}^d\to\mathbb{R}^{n-1}\times\mathbb{R}^1$$

be the standard wrinkled map with the wrinkle $S^{n-1}\subset\mathbb{R}^n\times\mathbb{R}^d$. Suppose that $n>1$. Then

(a) *there exists an embedding*

$$h\colon D^{n-1}\to \mathcal{O}p_{\mathbb{R}^{n+d}}\,D^n$$

and a framing μ of the normal bundle to $A=h(D^{n-1})\subset\mathbb{R}^n\times\mathbb{R}^d$ such that the pair (A,μ) forms a basis for a surgery eliminating the cusp $\Sigma^{11}(w)=S^{n-2}\subset S^{n-1}$ of the wrinkle;

(b) *if $d>0$ then one can arrange that the map $w(n+d,n,s)$ restricted to the union $\Sigma(w(n+d,n,s))\cup A$ is an embedding.*

Proof It is easy to construct an embedding $h=h_0$ and a framing μ to satisfy (a). The construction is clear from Figure 19. The manifold A in this case is obtained from the boundary of the upper semiball $\{|y|^2+z^2\le 1+\delta, z\ge 0\}\subset\mathbb{R}^{n-1}\times\mathbb{R}$ by removing the open disk $D^{n-1}=\{z=0,|y|<1\}$, and then smoothing the corner. Here $\delta>0$

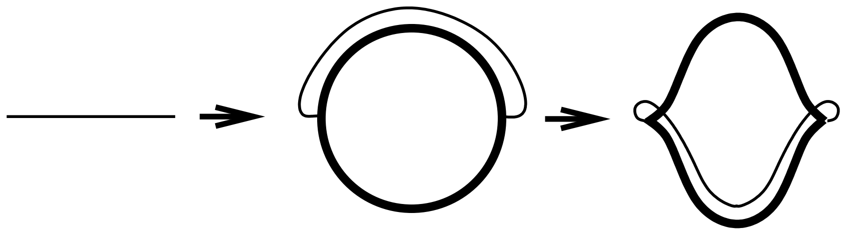


Figure 19: The embeddings h_0 and $g_0 = w \circ h_0$ (thin lines)

should be chosen small enough so that A lie in the prescribed neighborhood of the wrinkle. The framing μ is given by $\partial/\partial x_1, \dots, \partial/\partial x_d$ and the normal vector field to A in $\mathbb{R}^{n-1} \times \mathbb{R}$ which coincides with $\partial/\partial z$ near ∂A.

Unfortunately the embedding h_0 does not satisfies property (b). However, if $d > 0$ this can be corrected as follows. We suppose that the index $s > 0$ (if $s = 0$ then one should start with an embedding h_0 obtained by smoothing the boundary of the *lower* semiball). Let us denote by g_0 the composition $w \circ h_0$: $D^{n-1} \to \mathbb{R}^{n-1} \times \mathbb{R}$, and by g_0^{n-1} and g_0^1 the projections of g_0 to the first and second factors, respectively.

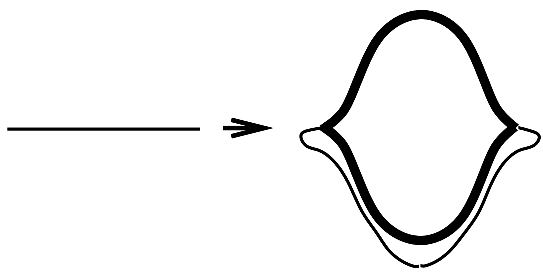


Figure 20: The embedding g

For any $\epsilon > 0$ one can choose δ in the construction of h_0 small enough to guarantee existence of a function α: $D^{n-1} \to [0, \epsilon)$ such that

- α vanishes along ∂D^{n-1} together with all its derivatives;
- $\alpha|_{\mathrm{Int}\, D^{n-1}} > 0$;
- the function $g^1 = g_0^1 - \alpha$ has a unique interior critical point, the minimum, at $0 \in D^{n-1}$;
- the map $g = (g_0^{n-1}, g^1)$: $D^{n-1} \to \mathbb{R}^{n-1} \times \mathbb{R}$ is an embedding, and the image $g(\mathrm{Int}\, D^{n-1})$ does not intersect the image of the wrinkle; see Figure 20.

Next, take an embedding $h\colon D^{n-1} \to \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^d$ given by

$$(y, z) = h_0(u), \quad x_1 = \sqrt{\alpha(u)}, \quad x_j = 0,\ j = 2, \dots, d,$$

$y \in \mathbb{R}^{n-1}$, $z \in \mathbb{R}$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. Then we have $g = w \circ h$, and hence the embedding h satisfies property (b) of Proposition 5.4. □

Recall Definition 2.3 of a special folded map. Combining Propositions 5.3 and 5.4 we get

Proposition 5.5 *Let $d > 0$. There exists a C^0–small perturbation of the map $w(n+d, n, s)|_{\mathcal{O}p_{\mathbb{R}^{n+d}} D^n}$ in an arbitrarily small neighborhood of the embedded disk $h(D^{n-1})$ constructed in Proposition 5.4 such that the resulting map $\widetilde{w}(n+d, d, s)$ is a special folded map with only one double fold (of index $s + 1/2$). Moreover, the regularized differentials of $w(n+d, n, s)$ and $\widetilde{w}(n+d, d, s)$ are homotopic.*

Theorem 5.1 and Proposition 5.5 yield Theorem 2.4.

5.2 Appendix B: Hurewicz theorem for oriented bordism

Recall that oriented bordism assigns to a pair (X, A) of spaces the groups $\Omega_n(X, A) = \Omega_n^{\mathrm{SO}}(X, A)$, defined as the set of bordism classes of continuous maps of pairs

$$f\colon (M^n, \partial M^n) \to (X, A)$$

for smooth oriented compact manifolds M^n with boundary ∂M. For A empty we write $\Omega_n(X) = \Omega_n(X, \varnothing)$. Ω_* is a "generalized homology theory", ie it satisfies the same formal properties as singular homology (the Eilenberg–Steenrod axioms except the dimension axiom). In particular, it is homotopy invariant, and there is a long exact sequence for pairs of spaces.

In this appendix we give a geometric proof of the following well known lemma.

Lemma 5.6 *Let $f\colon X \to Y$ be a continuous map of topological spaces. Then the following statements are equivalent.*

(i) *$f_*\colon H_k(X) \to H_k(Y)$ is an isomorphism for $k < n$ and an epimorphism for $k = n$.*

(ii) *$f_*\colon \Omega_k(X) \to \Omega_k(Y)$ is an isomorphism for $k < n$ and an epimorphism for $k = n$.*

In particular, f induces an isomorphism in homology in all degrees if and only if it does so in oriented bordism.

The starting point is the observation that for any base point $a \in A$, the Hurewicz map $\pi_n(X, A) \to H_n(X, A)$ factors as

$$\pi_n(X, A) \to \Omega_n(X, A) \to H_n(X, A),$$

where the first map sends the homotopy class of a map $(D^n, \partial D^n) \to (X, A)$ to its bordism class, and the second map sends the bordism class of a map $f\colon (M^n, \partial M^n) \to (X, A)$ to the element $f_*([M]) \in H_n(X, A)$. We first prove a bordism version of Whitehead's theorem.

Lemma 5.7 *If (Y, A) is an $(n-1)$–connected pair, then the maps above induce isomorphisms $\pi_n(Y, A) \cong \Omega_n(Y, A) \cong H_n(Y, A)$.*

Proof of Lemma 5.7 The classical Whitehead theorem says that the composite is an isomorphism, so it suffices to prove that $\pi_n(Y, A) \to \Omega_n(Y, A)$ is surjective. If $f\colon (M, \partial M) \to (Y, A)$ is a representative, we can first assume M is path connected and pick a CW structure with only one n–cell. Let $e\colon D^n \to M$ the characteristic map of that cell. By induction on cells, we can use that (Y, A) is $(n-1)$–connected to homotope f to a map $g\colon (M, M^{n-1}) \to (Y, A)$ that maps $(n-1)$–skeleton into A. Then f is cobordant to the map $e \circ g\colon (D^n, \partial D^n) \to (Y, A)$ which (after a further homotopy to make it basepoint preserving) represents an element of $\pi_n(Y, A)$. □

Proof of Lemma 5.6 Both H_* and Ω_* are homotopy invariant, so we can use mapping cylinders to reduce to the case where $f\colon X \to Y$ is the inclusion of a subspace $A \subset Y$. Using the long exact sequence for H_* and Ω_* we must prove that $H_k(Y, A) = 0$ for all $k \le n$ if and only if $\Omega_k(Y, A)$ for all $k \le n$. We first treat the case where Y and A are simply connected. This case follows from Lemma 5.7 and induction on n.

Ordinary homology satisfies $H_{k+1}(\Sigma Y, \Sigma A) = H_k(Y, A)$, where Σ denotes the unreduced suspension. The proof of this fact uses only the axioms, so in the same way we get $\Omega_{k+1}(\Sigma Y, \Sigma A) = H_k(Y, A)$. The general case then follows by observing that the double suspension of any space is simply connected. □

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