THE HOMOTOPY TYPE OF THE COBORDISM CATEGORY

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Abstract. The embedded cobordism category under study in this paper generalizes the category of conformal surfaces, introduced by G. Segal in [Seg04] in order to formalize the concept of field theories. Our main result identifies the homotopy type of the classifying space of the embedded d-dimensional cobordism category for all d. For $d = 2$, our results lead to a new proof of the generalized Mumford conjecture, somewhat different in spirit from the original one, presented in [MW02].

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1. Introduction and results

The conformal surface category $\mathcal{S}$ is defined as follows. For each non-negative integer m there is one object C_m of $\mathcal{S}$, namely the 1-manifold $S^1 \times \{1, 2, \dots, m\}$.

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A morphism from C_m to C_n is an isomorphism class of a Riemann surface Σ with boundary $\partial\Sigma$ together with an orientation-preserving diffeomorphism $\partial\Sigma \to C_n \amalg -C_m$. The composition is by sewing surfaces together.

Given a differentiable subsurface $F \subseteq [a_0, a_1] \times \mathbb{R}^{n+1}$ with $\partial F = F \cap \{a_0, a_1\} \times \mathbb{R}^{n+1}$, each tangent space T_pF inherits an inner product from the surrounding euclidean space and hence a conformal structure. If F is oriented, this induces a complex structure on F. The category $\mathcal{C}_2$ of embedded surfaces can thus be viewed as a substitute for the conformal surface category. It is a consequence of Teichmüller theory that their classifying spaces are rationally homotopy equivalent.

The embedded surface category has an obvious generalization to higher dimensions. For any $d \geq 0$, we have a category $\mathcal{C}_d$ whose morphisms are d-dimensional submanifolds $W \subseteq [a_0, a_1] \times \mathbb{R}^{n+d-1}$ that intersect the walls $\{a_0, a_1\} \times \mathbb{R}^{n+d-1}$ transversely in ∂W. The codimension n is arbitrarily large, and not part of the structure. Viewing W as a morphism from the incoming boundary $\partial_{\text{in}} W = \{a_0\} \times \mathbb{R}^{n+d-1} \cap W$ to the outgoing boundary $\partial_{\text{out}} W = \{a_1\} \times \mathbb{R}^{n+d-1} \cap W$, and using union as composition, we get the embedded cobordism category $\mathcal{C}_d$.

It is a topological category in the sense that the total set of objects and the total set of morphisms have topologies such that the structure maps (source, target, identity and composition) are continuous. In fact, there are homotopy equivalences

$$\operatorname{ob}\mathcal{C}_d \simeq \coprod_M B\operatorname{Diff}(M), \quad \operatorname{mor}\mathcal{C}_d \simeq \coprod_W B\operatorname{Diff}(W; \{\partial_{\text{in}}\}, \{\partial_{\text{out}}\})$$

where M varies over closed $(d-1)$-dimensional manifolds and W over d-dimensional cobordisms, one in each diffeomorphism class. Here $\operatorname{Diff}(M)$ denotes the topological group of diffeomorphisms of M and $\operatorname{Diff}(W, \{\partial_{\text{in}}\}, \{\partial_{\text{out}}\})$ denotes the group of diffeomorphisms of W that restrict to diffeomorphisms of the incoming and outgoing boundaries. Source and target maps are induced by restriction.

In order to describe our main result about the homotopy type of the classifying space $B\mathcal{C}_d$, we need some notation. Let $G(d, n)$ denote the Grassmannian of d-dimensional linear subspaces of $\mathbb{R}^{n+d}$. There are two standard vector bundles, $U_{d,n}$ and $U_{d,n}^{\perp}$, over $G(d, n)$. We are interested in the n-dimensional one with total space

$$U_{d,n}^{\perp} = \{(V, v) \in G(d, n) \times \mathbb{R}^{d+n} \mid v \perp V\}.$$

The Thom spaces (one-point compactifications) $\operatorname{Th}(U_{d,n}^{\perp})$ define a spectrum $MTO(d)$ as n varies*. The $(n + d)$th space in the spectrum $MTO(d)$ is $\operatorname{Th}(U_{d,n}^{\perp})$. We are

*This convenient and flexible notation was suggested by Mike Hopkins. $O(d)$ is the structure group for Tangent bundles of manifolds, as opposed to the standard notation $MO(d)$ for the Thom space of $U_{d,\infty} \to G(d, \infty)$, where $O(d)$ is the structure group for normal bundles of manifolds.

primarily interested in the direct limit

$$\Omega^{\infty-1} MTO(d) = \operatorname*{colim}_{n\to\infty} \Omega^{n+d-1} \operatorname{Th}(U_{d,n}^{\perp}).$$

$MTO(d)$ and $\Omega^{\infty-1}MTO(d)$ are described in more detail in section 3.1.

Given a morphism $W \subseteq [a_0, a_1] \times \mathbb{R}^{n+d-1}$, the Pontrjagin-Thom collapse map onto a tubular neighborhood gives a map

$$[a_0, a_1]_+ \wedge S^{n+d-1} \to \operatorname{Th}(U_{d,n}^{\perp}),$$

whose adjoint determines a path in $\Omega^{\infty-1}MTO(d)$ as $n \to \infty$. With more care, one gets a functor from $\mathcal{C}_d$ to the category $\operatorname{Path}(\Omega^{\infty-1}MTO(d))$, whose objects are points in $\Omega^{\infty-1}MTO(d)$ and whose morphisms are continuous paths.

The classifying space of a path category is always homotopy equivalent to the underlying space. We therefore get a map

$$\alpha : B\mathcal{C}_d \to \Omega^{\infty-1}MTO(d) \tag{1.1}$$

(cf. [MT01] for $d = 2$).

Main Theorem. *The map $\alpha : B\mathcal{C}_d \to \Omega^{\infty-1}MTO(d)$ is a weak homotopy equivalence.*

For any category C, the set of components $\pi_0 BC$ can be described as the quotient of the set $\pi_0 \operatorname{ob}(C)$ by the equivalence relation generated by the morphisms. For the category $\mathcal{C}_d$, this gives that $\pi_0 B\mathcal{C}_d$ is the group Ω_{d-1}^{O} of cobordism classes of closed unoriented manifolds. As explained in section 3.1 below, the group of components $\pi_0\Omega^{\infty-1}MTO(d)$ is isomorphic to the homotopy group $\pi_{d-1}MO$ of the Thom spectrum MO. Thus the main theorem can be seen as generalization of Thom's theorem: $\Omega_{d-1}^{O} \cong \pi_{d-1}MO$.

More generally we also consider the cobordism category $\mathcal{C}_\theta$ of manifolds with tangential structure, given by a lifting of the classifying map for the tangent bundle over a fibration $\theta : B \to G(d, \infty)$. In this case, the right hand side of (1.1) gets replaced by a spectrum $MT(\theta)$ whose $(n+d)$th space is $\operatorname{Th}(\theta^* U_{d,n}^{\perp})$. Chapter 5 defines $\mathcal{C}_\theta$ and $MT(\theta)$ in more detail, and proves the following version of the main theorem.

Main Theorem (with tangential structures). *There is a weak homotopy equivalence $\alpha^\theta : B\mathcal{C}_\theta \to \Omega^{\infty-1}MT(\theta)$.*

The simplest example of a tangential structure is that of an ordinary orientation, leading to the category $\mathcal{C}_d^+$ of oriented embedded cobordisms. In this case, the target of α becomes the oriented version $\Omega^{\infty-1}MTSO(d)$, which differs from $\Omega^{\infty-1}MTO(d)$ only in that we start with the Grassmannian $G^+(d, n)$ of oriented d-planes in $\mathbb{R}^{n+d}$. Another interesting special case leads to the category $\mathcal{C}_d^+(X)$ of oriented manifolds with a continuous map to a background space X. In this case our result is a weak equivalence

$$B\mathcal{C}_d^+(X) \simeq \Omega^{\infty-1}(MTSO(d) \wedge X_+).$$

In particular, the homotopy groups $\pi_* B\mathcal{C}_d^+(X)$ becomes a generalized homology theory as a functor of the background space X, with coefficients $\pi_*\Omega^{\infty-1}MTSO(d)$. The same works in the non-oriented situation.

We shall write $MT(d) = MTO(d)$ and $MT(d)^+ = MTSO(d)$ for brevity, since we are mostly concerned with these two cases.

For any topological category $\mathcal{C}$ and objects $x, y \in \operatorname{ob}\mathcal{C}$, there is a continuous map

$$\mathcal{C}(x, y) \to \Omega_{x,y} B\mathcal{C},$$

from the space of morphisms in $\mathcal{C}$ from x to y to the space $\Omega_{x,y}B\mathcal{C}$ of paths in $B\mathcal{C}$ from x to y. In the case of the oriented cobordism category we get for every oriented d-manifold W a map

$$\sigma : B\operatorname{Diff}^+(W; \partial W) \to \Omega B\mathcal{C}_d^+$$

into the loop space of $B\mathcal{C}_d^+$. For $d = 2$ and $W = W_{g,n}$ an oriented surface of genus g,

$$B\operatorname{Diff}^+(W, \partial W) \simeq B\Gamma_{g,n},$$

where $\Gamma_{g,n} = \pi_0 \operatorname{Diff}^+(W, \partial W)$ is the mapping class group of W. In this case, the composition

$$B\Gamma_{\infty,n} \to \Omega_0 B\mathcal{C}_2^+ \xrightarrow{\simeq} \Omega_0^\infty MT(2)^+$$

induces an isomorphism in integral homology. This is the generalized Mumford conjecture, proved in [MW02]. We give a new proof of this below, based on the above Main Theorem.

2. The cobordism category and its sheaves

2.1. **The cobordism category.** We fix the integer $d \geq 0$. The objects of the d-dimensional cobordism category $\mathcal{C}_d$ are closed $(d-1)$-dimensional submanifolds of high-dimensional euclidean space; the morphisms are d-dimensional embedded cobordisms with a collared boundary.

More precisely, an object of $\mathcal{C}_d$ is a pair (M, a) with $a \in \mathbb{R}$, and such that M is a closed $(d-1)$-dimensional submanifold

$$M \subseteq \mathbb{R}^{d-1+\infty} \quad , \quad \mathbb{R}^{d-1+\infty} = \operatorname*{colim}_{n\to\infty} \mathbb{R}^{d-1+n}$$

A non-identity morphism from (M_0, a_0) to (M_1, a_1) is a triple (W, a_0, a_1) consisting of the numbers a_0, a_1, which must satisfy $a_0 < a_1$, and a d-dimensional compact submanifold

$$W \subseteq [a_0, a_1] \times \mathbb{R}^{d-1+\infty},$$

such that for some $\varepsilon > 0$ we have

(i) $W \cap ([a_0, a_0 + \varepsilon) \times \mathbb{R}^{d-1+\infty}) = [a_0, a_0 + \varepsilon) \times M_0$,
(ii) $W \cap ((a_1 - \varepsilon, a_1] \times \mathbb{R}^{d-1+\infty}) = (a_1 - \varepsilon, a_1] \times M_1$,
(iii) $\partial W = W \cap (\{a_0, a_1\} \times \mathbb{R}^{d-1+\infty})$.

Composition is union of subsets (of $\mathbb{R} \times \mathbb{R}^{d-1+\infty}$):

$$(W_1, a_0, a_1) \circ (W_2, a_1, a_2) = (W_1 \cup W_2, a_0, a_2).$$

This defines $\mathcal{C}_d$ as a category of sets. We describe its topology.

Given a closed smooth $(d-1)$-manifold M, let $\operatorname{Emb}(M, \mathbb{R}^{d-1+n})$ denote the space of smooth embeddings, and write

$$\operatorname{Emb}(M, \mathbb{R}^{d-1+\infty}) = \operatorname*{colim}_{n\to\infty} \operatorname{Emb}(M, \mathbb{R}^{d-1+n}).$$

Composing an embedding with a diffeomorphism of M gives a free action of $\operatorname{Diff}(M)$ on the embedding space, and the orbit map

$$\operatorname{Emb}(M, \mathbb{R}^{d-1+\infty}) \to \operatorname{Emb}(M, \mathbb{R}^{d-1+\infty}) / \operatorname{Diff}(M)$$

is a principal $\operatorname{Diff}(M)$ bundle in the sense of [Ste51], if $\operatorname{Emb}(M, \mathbb{R}^{d-1+\infty})$ and $\operatorname{Diff}(M)$ are given Whitney C^∞ topology.

Let $E_\infty(M) = \operatorname{Emb}(M, \mathbb{R}^{d-1+\infty}) \times_{\operatorname{Diff}(M)} M$ and let $B_\infty(M)$ be the orbit space $\operatorname{Emb}(M, \mathbb{R}^{d-1+\infty})/\operatorname{Diff}(M)$. The associated fiber bundle

$$E_\infty(M) \to B_\infty(M) \tag{2.1}$$

has fiber M and structure group $\operatorname{Diff}(M)$. By Whitney's embedding theorem $\operatorname{Emb}(M, \mathbb{R}^{d-1+\infty})$ is contractible, so $B_\infty(M) \simeq B\operatorname{Diff}(M)$. In [KM97] a convenient category of infinite dimensional manifolds is described in which $\operatorname{Diff}(M)$ is a Lie group and (2.1) is a smooth fiber bundle. The fiber bundle (2.1) comes with a natural embedding $E_\infty(M) \subset B_\infty(M) \times \mathbb{R}^{d-1+\infty}$. With this structure, it is universal. More precisely, if $f : X \to B_\infty(M)$ is a smooth map from a smooth manifold X^d, then the pullback

$$f^*(E_\infty(M)) = \{(x, v) \in X \times \mathbb{R}^{d-1+\infty} \mid (f(x), v) \in E_\infty(M)\}$$

is a smooth $(k+d)$-dimensional submanifold $E \subseteq X \times \mathbb{R}^{d-1+\infty}$ such that the projection $E \to X$ is a smooth fiber bundle with fiber M. Any such $E \subseteq X \times \mathbb{R}^{d-1+\infty}$ is induced by a unique smooth map $f : X \to B_\infty(M)$.

Now the set of objects of $\mathcal{C}_d$ is

$$\operatorname{ob} \mathcal{C}_d \cong \mathbb{R} \times \coprod_M B_\infty(M), \tag{2.2}$$

where M varies over closed $(d-1)$-manifolds, one in each diffeomorphism class. We use this identification to topologize $\operatorname{ob} \mathcal{C}_d$.

The set of morphisms in $\mathcal{C}_d$ is topologized in a similar fashion. Let (W, h_0, h_1) be an abstract cobordism from M_0 to M_1, i.e. a triple consisting of a smooth compact d-manifold W and embeddings ("collars")

$$\begin{aligned} h_0 &: [0,1) \times M_0 \to W \\ h_1 &: (0,1] \times M_1 \to W \end{aligned} \tag{2.3}$$

such that ∂W is the disjoint union of the two spaces $h_\nu(\{\nu\} \times M_\nu)$, $\nu = 0, 1$. For $0 < \varepsilon < \frac{1}{2}$, let $\mathrm{Emb}_\varepsilon(W, [0,1] \times \mathbb{R}^{d-1+n})$ be the space of embeddings

$$j : W \to [0,1] \times \mathbb{R}^{d-1+n}$$

for which there exists embeddings $j_\nu : M_\nu \to \mathbb{R}^{d-1+n}$, $\nu = 0, 1$, such that

$$j \circ h_0(t_0, x_0) = (t_0, j_0(x_0)) \quad \text{and} \quad j \circ h_1(t_1, x_1) = (t_1, j_1(x_1))$$

for all $t_0 \in [0, \varepsilon)$, $t_1 \in (1 - \varepsilon, 1]$, and $x_\nu \in M_\nu$. Let

$$\mathrm{Emb}(W, [0,1] \times \mathbb{R}^{d-1+\infty}) = \operatorname*{colim}_{\substack{n \to \infty \\ \varepsilon \to 0}} \mathrm{Emb}_\varepsilon(W, [0,1] \times \mathbb{R}^{d-1+n}).$$

Let $\mathrm{Diff}_\varepsilon(W)$ denote the group of diffeomorphisms of W that restrict to product diffeomorphisms on the ε-collars, and let $\mathrm{Diff}(W) = \mathrm{colim}_\varepsilon \mathrm{Diff}_\varepsilon(W)$.

As before, we get a principal $\mathrm{Diff}(W)$-bundle

$$\mathrm{Emb}(W, \mathbb{R}^{d-1+\infty}) \to \mathrm{Emb}(W, [0,1] \times \mathbb{R}^{d-1+\infty}) / \mathrm{Diff}(W),$$

and an associated fiber bundle

$$E_\infty(W) \to B_\infty(W) = \mathrm{Emb}(W, [0,1] \times \mathbb{R}^{d-1+\infty}) / \mathrm{Diff}(W)$$

with fiber W and structure group $\mathrm{Diff}(W)$, satisfying a universal property similar to the one for $E_\infty(M) \to B_\infty(M)$ described above.

Topologize $\mathrm{mor}\, \mathcal{C}_d$ by

$$\mathrm{mor}\, \mathcal{C}_d \cong \mathrm{ob}\, \mathcal{C}_d \amalg \coprod_W \mathbb{R}^2_+ \times B_\infty(W), \tag{2.4}$$

where $\mathbb{R}^2_+$ is the open half plane $a_0 < a_1$, and W varies over cobordisms $W = (W, h_0, h_1)$, one in each diffeomorphism class.

For $(a_0, a_1) \in \mathbb{R}^2_+$, let $l : [0,1] \to [a_0, a_1]$ be the affine map with $l(\nu) = a_\nu$, $\nu = 0, 1$. For an element $j \in \mathrm{Emb}_\varepsilon(W, [0,1] \times \mathbb{R}^{d-1+\infty})$ we identify the element $((a_0, a_1), [j]) \in \mathbb{R}^2_+ \times B_\infty(W)$ with the element $(a_0, a_1, E) \in \mathrm{mor}\, \mathcal{C}_d$, where E is the image

$$E = (l \circ j)(W) \subseteq [a_0, a_1] \times \mathbb{R}^{d-1+\infty}.$$

Let us point out a slight abuse of notation: Strictly speaking, we should include the collars h_0 and h_1 in the notation for the Emb and Diff spaces. Up to homotopy,

$$\mathrm{Diff}(W) \xrightarrow{\simeq} \mathrm{Diff}(W, \{\partial_{\mathrm{in}} W\}, \{\partial_{\mathrm{out}} W\}) \tag{2.5}$$

is the group of diffeomorphisms of W that restrict to diffeomorphisms of the incoming and of the outgoing boundary of the cobordism W.

Again, Whitney's embedding theorem implies that $B_\infty(W) \simeq B\,\mathrm{Diff}(W)$. With respect to this homotopy equivalence, composition in $\mathcal{C}_d$ is induced by the morphism of topological groups

$$\mathrm{Diff}(W_1) \times_{\mathrm{Diff}(M_1)} \mathrm{Diff}(W_2) \to \mathrm{Diff}(W),$$

where $\partial_{\mathrm{out}} W_1 = M_1 = \partial_{\mathrm{in}} W_2$, and $W = W_1 \cup_{M_1} W_2$.

Remark 2.1. (i) There is a reduced version $\widetilde{\mathcal{C}}_d$ where objects are embedded in $\{0\}\times\mathbb{R}^{d-1+\infty}$ and morphisms in $[0,a_1]\times\mathbb{R}^{d-1+\infty}$. The functor $\mathcal{C}_d\to\widetilde{\mathcal{C}}_d$ that maps a cobordism $W^d\subseteq[a_0,a_1]\times\mathbb{R}^{d-1+\infty}$ into $W^d-a_0\in[0,a_1-a_0]\times\mathbb{R}^{d-1+\infty}$ induces a homotopy equivalence on classifying spaces. Indeed, the nerves are related by a pullback diagram

$$\begin{tikzcd} N_k\mathcal{C}_d \arrow[r] \arrow[d] & N_k\widetilde{\mathcal{C}}_d \arrow[d] \\ N_k(\mathbb{R},\le) \arrow[r] & N_k(\mathbb{R}_+,+) \end{tikzcd} \tag{2.6}$$

where $(\mathbb{R},\le)$ denotes $\mathbb{R}$ as an ordered set and $(\mathbb{R}_+,+)$ denotes $\mathbb{R}_+=\{0\}\amalg(0,\infty)$ as a monoid under addition. The two vertical maps are fibrations, and the bottom horizontal map is a weak equivalence. Therefore the functor $\mathcal{C}_d\to\widetilde{\mathcal{C}}_d$ induces a levelwise homotopy equivalence on nerves.
(ii) In the previous remark it is crucial that $\mathbb{R}$ be given its usual topology. More precisely, let $\mathbb{R}^\delta$ denote $\mathbb{R}$ with the discrete topology, and define $\mathcal{C}_d^\delta$ and $\widetilde{\mathcal{C}}_d^\delta$ using $\mathbb{R}^\delta$ instead of $\mathbb{R}$ in the homeomorphisms (2.2) and (2.4). Then the right hand vertical map in (2.6) defines a map $B\widetilde{\mathcal{C}}_d^\delta\to B(\mathbb{R}_+^\delta,+)$ which is a split surjection. By the group-completion theorem [MS76], $\pi_1B(\mathbb{R}_+^\delta,+)\cong\mathbb{R}$, and this is a direct summand of $\pi_1B\widetilde{\mathcal{C}}_d^\delta$, so the main theorem fails for $\widetilde{\mathcal{C}}_d^\delta$. We shall see later that $B\mathcal{C}_d^\delta\to B\mathcal{C}_d$ is a homotopy equivalence (cf. Remark 4.5).
(iii) There is a version $\mathcal{C}_d^+$ of $\mathcal{C}_d$ where one adds an orientation to the objects and morphisms in the usual way. For $d=2$, the reduced version $\widetilde{\mathcal{C}}_d^+$ is the surface category $\mathcal{Y}$ of [MT01, §2].

2.2. **Recollection from** [MW02] **on sheaves.** Let $\mathcal{X}$ denote the category of smooth manifolds without boundary and smooth maps. We shall consider sheaves on $\mathcal{X}$, that is, contravariant functors $\mathcal{F}$ on $\mathcal{X}$ that satisfy the sheaf condition: for any open covering $\mathcal{U}=\{U_j\mid j\in J\}$ of an object X in $\mathcal{X}$ and elements $s_j\in\mathcal{F}(U_j)$ with $s_j|U_i\cap U_j=s_i|U_i\cap U_j$ there is a unique $s\in\mathcal{F}(X)$ that restricts to s_j for all j. We have the Yoneda embedding of $\mathcal{X}$ into the category $\mathrm{Sh}(\mathcal{X})$ of sheaves on $\mathcal{X}$ that to $X\in\mathcal{X}$ associates the representable sheaf $\tilde{X}=C^\infty(-,X)\in\mathrm{Sh}(\mathcal{X})$.

For the functors $\mathcal{F}$ we shall consider, $\mathcal{F}(X)$ consists of spaces over X with extra properties. In general the set of spaces E over X is not a functor under pull-back ($(g\circ f)^*(E)\neq f^*(g^*(E))$). But if $E\to X$ comes from subsets $E\subseteq X\times U$ where U is some "universe" then pull-backs with respect to $f\colon X'\to X$ in $\mathcal{X}$, defined as

$$f^*(E)=\{(x',u)\mid (f(x'),u)\in E\}\subseteq X'\times U,$$

is a functorial construction.

A set-valued sheaf $\mathcal{F}$ on $\mathcal{X}$ gives rise to a *representing space* $|\mathcal{F}|$, constructed as the topological realization of the following simplicial set. The hyperplane (open or extended simplex)

$$\Delta^\ell_e = \{(t_0, \ldots, t_\ell) \in \mathbb{R}^{\ell+1} \mid \sum t_i = 1\}$$

is an object of $\mathcal{X}$, and

$$[\ell] \longmapsto \mathcal{F}(\Delta^\ell_e)$$

is a simplicial set. The space $|\mathcal{F}|$ is its standard topological realization. This is a representing space in the following sense.

Definition 2.2. Two elements $s_0, s_1 \in \mathcal{F}(X)$ are concordant if there exists an $s \in \mathcal{F}(X \times \mathbb{R})$ which agrees with $\mathrm{pr}^*(s_0)$ near $X \times (-\infty, 0]$ and with $\mathrm{pr}^*(s_1)$ near $X \times [1, +\infty)$ where $\mathrm{pr}\colon X \times \mathbb{R} \to X$ is the projection.

The set of concordance classes will be denoted by $\mathcal{F}[X]$. The space $|\mathcal{F}|$ above is a representing space in the sense that $\mathcal{F}[X]$ is in bijective correspondence with the set of homotopy classes of continuous maps from X into $|\mathcal{F}|$:

$$\mathcal{F}[X] \cong [X, |\mathcal{F}|] \tag{2.7}$$

by Proposition A.1.1 of [MW02]. We describe the map. For $\tilde{X} = C^\infty(-, X)$, $[l] \mapsto \tilde{X}(\Delta^l_e)$ is the (extended, smooth) total simplicial set of X, and satisfies that the canonical map $|\tilde{X}| \to X$ is a homotopy equivalence ([Mil57]). An element $s \in \mathcal{F}(X)$ has an adjoint $\tilde{s} : \tilde{X} \to \mathcal{F}$, inducing $|\tilde{s}| : |\tilde{X}| \to |\mathcal{F}|$, and thus a well defined homotopy class of maps $X \to |\mathcal{F}|$ which is easily seen to depend only on the concordance class of s.

Definition 2.3. A map $\tau\colon \mathcal{F}_1 \to \mathcal{F}_2$ is called a weak equivalence if the induced map from $|\mathcal{F}_1|$ to $|\mathcal{F}_2|$ induces an isomorphism on all homotopy groups.

There is a convenient criteria for deciding if a map of sheaves is a weak equivalence. This requires a relative version of Definition 2.2. Let $A \subseteq X$ be a closed subset of X, and let $s \in \operatorname{colim}_U \mathcal{F}(U)$ where U runs over open neighborhoods of A. Let $\mathcal{F}(X, A; s) \subseteq \mathcal{F}(X)$ be the subset of elements that agree with s near A.

Definition 2.4. Two elements $t_0, t_1 \in \mathcal{F}(X, A; s)$ are concordant rel. A if they are concordant by a concordance whose germ near A is the constant concordance of s. Let $\mathcal{F}_1[X, A; s]$ denote the set of concordance classes.

Criteria 2.5. *A map $\tau\colon \mathcal{F}_1 \to \mathcal{F}_2$ is a weak equivalence provided it induces a surjective map*

$$\mathcal{F}_1[X, A; s] \to \mathcal{F}_2[X, A; \tau(s)]$$

for all (X, A, s) as above.

Let $x_0 \in X$ and $s_0 \in \mathcal{F}(\{x_0\})$. This gives a germ $s_0 \in \operatorname{colim}_U \mathcal{F}(U)$ with U ranging over the open neighborhoods of x_0. There is the following relative version of (2.7), also proved in Appendix A of [MW02]: for every (X, A, s_0),

$$\mathcal{F}[X, A; s_0] \cong [(X, A), (|\mathcal{F}|, s_0)]$$

In particular the homotopy groups $\pi_n(|\mathcal{F}|, s_0)$ are equal to the relative concordance classes $\mathcal{F}[S^n, x_0; s_0]$. By Whitehead's theorem $\tau\colon \mathcal{F}_1 \to \mathcal{F}_2$ is a weak equivalence if and only if

$$\mathcal{F}_1[S^n, x_0; s_0] \xrightarrow{\cong} \mathcal{F}_2[S^n, x_0; \tau(s_0)]$$

is an equivalence for all basepoints x_0 and all $s_0 \in \mathcal{F}_1(x_0)$. This is sometimes a more convenient formulation than Criteria 2.5 above.

Actually, for the concrete sheaves we consider in this paper the representing spaces are "simple" in the sense of homotopy theory, and in this situation the base point $s_0 \in \mathcal{F}(*)$ is irrelevant: a map $\tau\colon \mathcal{F}_1 \to \mathcal{F}_2$ is a weak equivalence if and only if it induces a bijection $\mathcal{F}_1[X] \to \mathcal{F}_2[X]$ for all $X \in \mathcal{X}$. In fact, it suffices to check this when X is a sphere.

2.3. **A sheaf model for the cobordism category.** We apply the above to give a sheaf model of the cobordism category $\mathcal{C}_d$. First some notation. For functions $a_0, a_1\colon X \to \mathbb{R}$ with $a_0(x) \le a_1(x)$ at all $x \in X$, we write

$$X \times (a_0, a_1) = \{(x, u) \in X \times \mathbb{R} \mid a_0(x) < u < a_1(x)\}$$
$$X \times [a_0, a_1] = \{(x, u) \in X \times \mathbb{R} \mid a_0(x) \le u \le a_1(x)\}.$$

Given a submersion $\pi\colon W \to X$ of smooth manifolds (without boundary) and smooth maps

$$f\colon W \to \mathbb{R}, \quad a\colon X \to \mathbb{R},$$

we say that f is *fiberwise transverse* to a if the restriction f_x of f to $W_x = \pi^{-1}(x)$ is transversal to $a(x)$ for every $x \in X$, or equivalently if the graph $X \times \{a\}$ consists of regular values for $(\pi, f) : E \to X \times \mathbb{R}$. In this case

$$M = (f - a\pi)^{-1}(0) = \{z \in W \mid f(z) = a(\pi(z))\}$$

is a codimension one submanifold of W, and the restriction $\pi\colon M \to X$ is still a submersion.

For $X \in \mathcal{X}$ and smooth real functions

$$a_0 \le a_1\colon X \to \mathbb{R}, \quad \varepsilon\colon X \to (0, \infty)$$

we shall consider submanifolds

$$W \subseteq X \times (a_0 - \varepsilon, a_1 + \varepsilon) \times \mathbb{R}^{d-1+\infty}.$$

The three projections will be denoted

$$\pi\colon W \to X, \quad f\colon W \to \mathbb{R}, \quad j\colon W \to \mathbb{R}^{d-1+\infty}$$

unless otherwise specified.

Definition 2.6. For $X \in \mathcal{X}$ and smooth real functions $a_0 \le a_1$ and ε as above, the set $C_d^{\pitchfork}(X; a_0, a_1, \varepsilon)$ consists of all submanifolds

$$W \subseteq X \times (a_0 - \varepsilon, a_1 + \varepsilon) \times \mathbb{R}^{d-1+\infty}$$

which satisfies the following conditions:

(i) $\pi\colon W \to X$ is a submersion with d-dimensional fibers,
(ii) $(\pi, f)\colon W \to X \times (a_0 - \varepsilon, a_1 + \varepsilon)$ is proper,
(iii) The restriction of (π, f) to $(\pi, f)^{-1}(X \times (a_\nu - \varepsilon, a_\nu + \varepsilon))$ is a submersion for $\nu = 0, 1$.

The three conditions imply that $\pi\colon W \to X$ is a smooth fiber bundle rather than just a submersion. Indeed for each $\nu = 0, 1$, restricting (π, f) gives a map

$$(\pi, f)^{-1}(X \times (a_\nu - \varepsilon, a_\nu + \varepsilon)) \to X \times (a_\nu - \varepsilon, a_\nu + \varepsilon)$$

which is a proper submersion, and hence a smooth fiber bundle by Ehresmann's fibration lemma, cf. [BJ82, p. 84]. Similarly the restriction of π to

$$W[a_0, a_1] = W \cap X \times [a_0, a_1] \times \mathbb{R}^{d-1+\infty}$$

is a smooth fiber bundle with boundary. The result for $\pi\colon W \to X$ follows by gluing the collars.

We remove the dependence on ε and define

$$C_d^{\pitchfork}(X; a_0, a_1) = \operatorname*{colim}_{\varepsilon \to 0} C_d^{\pitchfork}(X; a_0, a_1, \varepsilon).$$

Definition 2.7. For $X \in \mathcal{X}$,

$$C_d^{\pitchfork}(X) = \coprod C_d^{\pitchfork}(X; a_0, a_1).$$

The disjoint union varies over all pairs of smooth functions with $a_0 \leq a_1$ and such that $\{x \mid a_0(x) = a_1(x)\}$ is open (hence a union of connected components of X). This defines a sheaf $C_d^{\pitchfork}$.

Taking union of embedded manifolds gives a partially defined map

$$C_d^{\pitchfork}(X; a_0, a_1) \times C_d^{\pitchfork}(X; a_1, a_2) \to C_d^{\pitchfork}(X; a_0, a_2)$$

and defines a category structure on $C_d^{\pitchfork}(X)$ with the objects (or identity morphisms) corresponding to $a_0 = a_1$.

A smooth map $\varphi\colon Y \to X$ induces a map of categories $\varphi^*\colon C_d^{\pitchfork}(X) \to C_d^{\pitchfork}(Y)$ by the pull-back construction of § 2.2: For

$$W \subseteq X \times (a_0 - \varepsilon, a_1 + \varepsilon) \times \mathbb{R}^{d-1+\infty},$$

$\varphi^* W = \{(y, u, r) \mid (\varphi(y), u, r) \in W\}$ is an element of $C_d^{\pitchfork}(Y; a_0\varphi, a_1\varphi, \varepsilon\varphi)$. This gives a **CAT**-valued sheaf

$$C_d^{\pitchfork}\colon \mathcal{X} \to \mathbf{CAT},$$

where **CAT** is the category of small categories.

An object of $C_d^{\pitchfork}(\mathrm{pt})$ is represented by a d-manifold $W \subseteq (a - \varepsilon, a + \varepsilon) \times \mathbb{R}^{d-1+\infty}$ such that $f : W \to (a - \varepsilon, a + \varepsilon)$ is a proper submersion. Thus $M = f^{-1}(0) \subseteq \mathbb{R}^{d-1+\infty}$ is a closed $(d-1)$-manifold. Only the *germ* of W near M is well-defined. As an abstract manifold, W is diffeomorphic to $M \times (a - \varepsilon, a + \varepsilon)$, but the embedding into $(a - \varepsilon, a + \varepsilon) \times \mathbb{R}^{d-1+\infty}$ need not be the product embedding. Hence the germ of W near M carries slightly more information than just the submanifold $M \subseteq \{a\} \times \mathbb{R}^{d-1+\infty}$. This motivates

Definition 2.8. Let $C_d(X; a_0, a_1, \varepsilon) \subseteq C_d^{\pitchfork}(X; a_0, a_1, \varepsilon)$ be the subset satisfying the further condition

(iv) For $x \in X$ and $\nu = 0, 1$, let J_ν be the interval $((a_\nu - \varepsilon)(x), (a_\nu + \varepsilon)(x)) \subseteq \mathbb{R}$, and let $V_\nu = (\pi, f)^{-1}(\{x\} \times J_\nu) \subseteq \{x\} \times J_\nu \times \mathbb{R}^{d-1+\infty}$. Then
$$V_\nu = \{x\} \times J_\nu \times M$$
for some $(d-1)$-dimensional submanifold $M \subseteq \mathbb{R}^{d-1+\infty}$.

Define $C_d(X; a_0, a_1) \subseteq C_d^{\pitchfork}(X; a_0, a_1)$ and $C_d(X) \subseteq C_d^{\pitchfork}(X)$ similarly.

It is easy to see that $C_d(X)$ is a full subcategory of $C_d^{\pitchfork}(X)$ and that
$$C_d \colon \mathcal{X} \to \mathbf{CAT}$$
is a sheaf of categories, isomorphic to the sheaf $C^\infty(-, \mathcal{C}_d)$, where $\mathcal{C}_d$ is equipped with the (infinite dimensional) smooth structure described in section 2.1. In particular we get a continuous functor
$$\eta : |C_d| \to \mathcal{C}_d.$$

Proposition 2.9. *$B\eta : B|C_d| \to B\mathcal{C}_d$ is a weak homotopy equivalence.*

Proof. The space $N_k|C_d|$ is the realization of the simplicial set
$$[l] \mapsto N_k C_d(\Delta_e^l) = C^\infty(\Delta_e^l, N_k\mathcal{C}).$$
A theorem from [Mil57] asserts that the realization of the singular simplicial set of any space Y is weakly homotopy equivalent to Y itself. This is also the case if one uses the extended simplices Δ_e^k to define the singular simplicial set, and for manifolds it is also true if we use smooth maps. This proves that the map
$$N_k\eta : N_k|C_d| \to N_k\mathcal{C}_d$$
is a weak homotopy equivalence for all k, and hence that $B\eta$ is a weak homotopy equivalence. □

2.4. **Cocycle sheaves.** We review the construction from [MW02, §4.1] of a model for the classifying space construction at the sheaf level.

Let $\mathcal{F}$ be any **CAT**-valued sheaf on $\mathcal{X}$. There is an associated set valued sheaf $\beta\mathcal{F}$. Choose, once and for all, an uncountable set J. An element of $\beta\mathcal{F}(X)$ is a pair $(\mathcal{U}, \Phi)$ where $\mathcal{U} = \{U_j \mid j \in J\}$ is a locally finite open cover of X, indexed by J, and Φ a certain collection of morphisms. In detail: given a non-empty finite subset $R \subseteq J$, let U_R be the intersection of the U_j's for $j \in R$. Then Φ is a collection $\varphi_{RS} \in N_1\mathcal{F}(U_S)$ indexed by pairs $R \subseteq S$ of non-empty finite subsets of J, subject to the conditions

(i) $\varphi_{RR} = \mathrm{id}_{c_R}$ for an object $c_R \in N_0\mathcal{F}(U_R)$,
(ii) For each non-empty finite $R \subseteq S$, φ_{RS} is a morphism from c_S to $c_R|U_S$,
(iii) For all triples $R \subseteq S \subseteq T$ of finite non-empty subsets of J, we have
$$\varphi_{RT} = (\varphi_{RS}|U_T) \circ \varphi_{ST}. \tag{2.8}$$

Theorem 4.1.2 of [MW02] asserts a weak homotopy equivalence

$$|\beta\mathcal{F}| \simeq B|\mathcal{F}|. \tag{2.9}$$

Remark 2.10. In the case $\mathcal{F}(X) = \operatorname{Map}(X, \mathcal{C})$ for some topological category $\mathcal{C}$ the construction $\beta\mathcal{F}$ takes the following form. Let $X_{\mathcal{U}}$ be the topological category from [Seg68]:

$$\operatorname{ob} X_{\mathcal{U}} = \coprod_{R} U_R \qquad\qquad \operatorname{mor} X_{\mathcal{U}} = \coprod_{R\subseteq S} U_S,$$

i.e. $X_{\mathcal{U}}$ is the topological poset of pairs (R, x), where $R \subseteq J$ is a finite non-empty subset and $x \in U_R$. If $R \subseteq S$ and $x = y$, then there is precisely one morphism $(S, x) \to (R, y)$, otherwise there is none.

Then (2.8) amounts to a continuous functor $\Phi \colon X_{\mathcal{U}} \to \mathcal{C}$. In general, (2.8) amounts to a functor $\tilde{X}_{\mathcal{U}} \to \mathcal{F}$, where $\tilde{X}_{\mathcal{U}} = C^\infty(-, X_{\mathcal{U}})$ is the (representable) sheaf of posets associated to $X_{\mathcal{U}}$.

A partition of unity $\{\lambda_j \mid j \in J\}$ subordinate to $\mathcal{U}$ defines a map from X to $BX_{\mathcal{U}}$ and Φ a map from $BX_{\mathcal{U}}$ to $B\mathcal{C}$. This induces a map

$$\beta\mathcal{F}[X] \to [X, B\mathcal{C}]$$

and (2.9) asserts that this is a bijection for all X.

3. The Thom spectra and their sheaves

3.1. **The spectrum $MT(d)$ and its infinite loop space.** We write $G(d, n)$ for the Grassmann manifold of d-dimensional linear subspaces of $\mathbb{R}^{d+n}$ and $G^+(d, n)$ for the double cover of $G(d, n)$ where the subspace is equipped with an orientation.

There are two distinguished vector bundles over $G(d, n)$, the tautological d-dimensional vector bundle $U_{d,n}$ consisting of pairs of a d-plane and a vector in that plane, and its orthogonal complement, the n-dimensional vector bundle $U_{d,n}^\perp$. The direct sum $U_{d,n} \oplus U_{d,n}^\perp$ is the product bundle $G(d, n) \times \mathbb{R}^{d+n}$.

The Thom spaces (one point compactifications) $\operatorname{Th}(U_{d,n}^\perp)$ form the spectrum $MT(d)$ as n varies. Indeed, since $U_{d,n+1}^\perp$ restricts over $G(d, n)$ to the direct sum of $U_{d,n}^\perp$ and a trivial line, there is an induced map

$$S^1 \wedge \operatorname{Th}(U_{d,n}^\perp) \to \operatorname{Th}(U_{d,n+1}^\perp). \tag{3.1}$$

The $(n + d)$th space of the spectrum $MT(d)$ is $\operatorname{Th}(U_{d,n}^\perp)$, and (3.1) provides the structure maps. The associated infinite loop space is therefore

$$\Omega^\infty MT(d) = \operatorname*{colim}_{n\to\infty} \Omega^{n+d} \operatorname{Th}(U_{d,n}^\perp),$$

where the maps in the colimit

$$\Omega^{n+d} \operatorname{Th}(U_{d,n}^\perp) \to \Omega^{n+d+1} \operatorname{Th}(U_{d,n+1}^\perp)$$

are the $(n + d)$-fold loops of the adjoints of (3.1).

There is a corresponding oriented version $MT(d)^+$ where one uses the Thom spaces of pull-backs $\theta^* U_{d,n}^\perp$, $\theta\colon G^+(d,n) \to G(d,n)$. The spectrum $MT(d)^+$ maps to $MT(d)$ and induces

$$\Omega^\infty MT(d)^+ \to \Omega^\infty MT(d).$$

Proposition 3.1. *There are homotopy fibration sequences*

$$\Omega^\infty MT(d) \to \Omega^\infty \Sigma^\infty (BO(d)_+) \xrightarrow{\partial} \Omega^\infty MT(d-1),$$
$$\Omega^\infty MT(d)^+ \to \Omega^\infty \Sigma^\infty (BSO(d)_+) \xrightarrow{\partial} \Omega^\infty MT(d-1)^+.$$

Proof. For any two vector bundles E and F over the same base B there is a cofiber sequence

$$\mathrm{Th}(p^*E) \to \mathrm{Th}(E) \to \mathrm{Th}(E \oplus F) \tag{3.2}$$

where $p\colon S(F) \to X$ is the bundle projection of the sphere bundles.

Apply this to $X = G(d,n)$, $E = U_{d,n}^\perp$, $F = U_{d,n}$. The sphere bundle is

$$S(U_{d,n}) = O(n+d)/O(n) \times O(d-1).$$

Since $G(d-1,n) = O(n+d-1)/O(n) \times O(d-1)$, the natural map $G(d-1,n) \to S(U_{d,n})$ is $(n+d-2)$-connected. The bundle $p^* U_{d,n}^\perp$ over $S(U_{d,n})$ restricts to $U_{d-1,n}^\perp$ over $G(d-1,n)$, so

$$\mathrm{Th}(U_{d-1,n}^\perp) \to \mathrm{Th}(p^* U_{d,n}^\perp)$$

is $(2n+d-2)$-connected. The right-hand term in (3.2) is $G(d,n)_+ \wedge S^{n+d}$, and the map $G(d,n) \to BO(d)$ is $(n-1)$-connected ($BO(d) = G(d,\infty)$).

The cofiber sequence (3.2) gives a cofiber sequence of spectra

$$\Sigma^{-1} MT(d-1) \to MT(d) \to \Sigma^\infty (BO(d)_+) \to MT(d-1) \tag{3.3}$$

and an associated homotopy fibration sequence

$$\Omega^\infty MT(d) \to \Omega^\infty \Sigma^\infty (BO(d)_+) \to \Omega^\infty MT(d-1)$$

of infinite loop spaces. The oriented case is completely similar. □

Remark 3.2. For $d = 1$, the sequences in Proposition 3.1 are

$$\Omega^\infty MT(1) \to \Omega^\infty \Sigma^\infty (\mathbb{R}P_+^\infty) \xrightarrow{\partial} \Omega^\infty \Sigma^\infty$$
$$\Omega^\infty MT(1)^+ \to \Omega^\infty \Sigma^\infty \xrightarrow{\partial} \Omega^\infty \Sigma^\infty \times \Omega^\infty \Sigma^\infty.$$

In the first sequence, ∂ is the stable transfer associated with the universal double covering space. In the oriented case, ∂ is the diagonal. Thus

$$\Omega^\infty MT(1) = \Omega^\infty \mathbb{R}P_{-1}^\infty, \quad \Omega^\infty MT(1)^+ = \Omega(\Omega^\infty \Sigma^\infty).$$

The oriented Grassmannian $G^+(2,\infty)$ is homotopy equivalent to $\mathbb{C}P^\infty$, and the space $\Omega^\infty MT(2)^+$ is homotopy equivalent to the space $\Omega^\infty \mathbb{C}P_{-1}^\infty$, in the notation from [MW02].

The cofiber sequence (3.3) defines a direct system of spectra

$$MT(0) \to \Sigma MT(1) \to \cdots \to \Sigma^{d-1} MT(d-1) \to \Sigma^d MT(d) \to \cdots \qquad (3.4)$$

whose direct limit is the universal Thom spectrum usually denoted MO. The homotopy groups of MO form the unoriented bordism ring

$$\pi_{d-1} MO = MO_{d-1}(\mathrm{pt}) = \Omega^O_{d-1}.$$

The direct system (3.4) can be thought of as a filtration of MO, with filtration quotients $\Sigma^d BO(d)_+$. In particular, the maps in the direct system induce an isomorphism

$$\pi_{-1} MT(d) = \pi_{d-1} \Sigma^d MT(d) \xrightarrow{\cong} \pi_{d-1} MO = \Omega^O_{d-1},$$

and an exact sequence

$$\pi_0 MT(d+1) \xrightarrow{\chi} \mathbb{Z} \xrightarrow{S^d} \pi_0 MT(d) \to \Omega^O_d \to 0. \qquad (3.5)$$

The map $\chi : \pi_0 MT(d+1) \to \mathbb{Z}$ corresponds under the homotopy equivalence of our main theorem to the map that to a closed $(d+1)$-manifold W, thought of as an endomorphism in $\mathcal{C}_{d+1}$ of the empty d-manifold, associates the Euler characteristic $\chi(W) \in \mathbb{Z}$. The map $S^d : \mathbb{Z} \to \pi_0 MT(d)$ corresponds to the d-sphere S^d, thought of as an endomorphism in $\mathcal{C}_d$ of the empty $(d-1)$-manifold. For odd d, χ is surjective ($\chi(\mathbb{R}P^{d+1}) = 1$), so the sequence (3.5) defines an isomorphism $\pi_0 MT(d) \cong \Omega^O_d$. On the other hand $\chi = 0$ for even d by Poincaré duality, so the sequence (3.5) works out to be

$$0 \to \mathbb{Z} \xrightarrow{S^d} \pi_0 MT(d) \to \Omega^O_d \to 0.$$

3.2. **Using Phillips' submersion theorem.** We give a sheaf model for the space $\Omega^{\infty-1} MT(d)$.

Definition 3.3. For a natural number $n > 0$ and $X \in \mathcal{X}$, an element of $D_d(X; n)$ is a submanifold

$$W \subseteq X \times \mathbb{R} \times \mathbb{R}^{d-1+n},$$

with projections π, f, and j, respectively, such that

(i) $\pi \colon W \to X$ is a submersion with d-dimensional fibers.
(ii) $(\pi, f) \colon W \to X \times \mathbb{R}$ is proper.

This defines a set valued sheaf $D_d(-; n) \in \mathrm{Sh}(\mathcal{X})$. Let D_d be the colimit (in $\mathrm{Sh}(\mathcal{X})$) of $D_d(-; n)$ as $n \to \infty$. Explicitly, $D_d(X)$ is the set of submanifolds $W \subseteq X \times \mathbb{R} \times \mathbb{R}^{d-1+\infty}$ satisfying (i) and (ii) above, and such that for each compact $K \subseteq X$ there exists an n with $\pi^{-1}(K) \subseteq K \times \mathbb{R} \times \mathbb{R}^{d-1+n}$.

We will prove the following theorem by constructing a natural bijection $[X, \Omega^{\infty-1} MT(d)] \cong D_d[X]$.

Theorem 3.4. *There is a weak homotopy equivalence*

$$|D_d| \xrightarrow{\simeq} \Omega^{\infty-1} MT(d).$$

Given $W \subseteq X \times \mathbb{R} \times \mathbb{R}^{d-1+n}$ with n-dimensional normal bundle $N \to W$, there is a vector bundle map

$$\begin{tikzcd} N \arrow[r, "\hat{\gamma}"] \arrow[d] & U_{d,n}^{\perp} \arrow[d] \\ W \arrow[r, "\gamma"'] & G(d,n). \end{tikzcd} \tag{3.6}$$

Write W_x for the intersection $W_x = W \cap \{x\} \times \mathbb{R} \times \mathbb{R}^{d-1+n}$. Then $\gamma(z) = T_z(W_{\pi(z)})$, considered as a subspace of $\mathbb{R}^{d+n}$. The normal fiber N_z of W in $X \times \mathbb{R} \times \mathbb{R}^{d-1+n}$ is the normal fiber of W_x in $\mathbb{R}^{d+n}$, so is equal to $\gamma(z)^{\perp}$; this defines $\hat{\gamma}$ in (3.6).

Next we pick a regular value for $f\colon W \to \mathbb{R}$, say $0 \in \mathbb{R}$, and let $M = f^{-1}(0)$. Then the normal bundle N of $W \subseteq X \times \mathbb{R} \times \mathbb{R}^{d-1+n}$ restricts to the normal bundle of $M \subset X \times \mathbb{R}^{d-1+n}$. Choose a tubular neighborhood of M in $X \times \mathbb{R}^{d-1+n}$, and let

$$e\colon N|M \to X \times \mathbb{R}^{d-1+n}$$

be the associated embedding ([BJ82, §12]). The induced map of one-point compactifications, composed with (3.6), gives a map

$$g\colon X_+ \wedge S^{d-1+n} \to \mathrm{Th}(U_{d,n}^{\perp}) \tag{3.7}$$

whose homotopy class is independent of the choices made (when $n \gg d$). Its adjoint is a well-defined homotopy class of maps from X to $\Omega^{\infty-1}MT(d)$. This defines

$$\rho\colon D_d[X] \to [X, \Omega^{\infty-1}MT(d)].$$

We now construct an inverse to ρ using transversality and Phillips' submersion theorem. We give the argument only in the case where X is compact. Any map (3.7) is homotopic to a map that is transversal to the zero section, and

$$M = g^{-1}(G(d,n)) \subseteq X \times \mathbb{R}^{d-1+n}$$

is a submanifold. The projection $\pi_0\colon M \to X$ is proper, and the normal bundle is $N = g^*(U_{d,n}^{\perp})$. Define $T^{\pi}M = g^*(U_{d,n})$ so that

$$N \oplus T^{\pi}M = M \times \mathbb{R}^{n+d}.$$

Combined with the bundle information of the embedding of M in $X \times \mathbb{R}^{d-1+n}$ this yields an isomorphism of vector bundles over M

$$TM \times \mathbb{R}^{n+d} \xrightarrow{\cong} (\pi_0^*TX \oplus T^{\pi}M) \times \mathbb{R}^{d-1+n}. \tag{3.8}$$

By standard obstruction theory (cf. [MW02], Lemma 3.2.3) there is an isomorphism (unique up to concordance)

$$\hat{\pi}_0\colon TM \times \mathbb{R} \xrightarrow{\cong} \pi_0^*TX \oplus T^{\pi}M$$

that induces (3.8). Set $W = M \times \mathbb{R}$, $\pi_1 = \pi_0 \circ \mathrm{pr}_M$ and $T^\pi W = \mathrm{pr}_M^* T^\pi M$. Then

$$TW \xrightarrow{\cong} \pi_1^* TX \oplus T^\pi W, \tag{3.9}$$

and since W has no closed components we are in a position to apply the submersion theorem. Indeed, (3.9) gives a bundle epimorphism $\hat{\pi}_1 \colon TW \to TX$ over $\pi_1 \colon W \to X$. By Phillips' theorem, there is a homotopy $(\pi_t, \hat{\pi}_t), t \in [1,2]$ through bundle epimorphisms, from $(\pi_1, \hat{\pi}_1)$ to a pair $(\pi_2, d\pi_2)$, i.e. to a submersion π_2. Let $f : W \to \mathbb{R}$ be the projection. Then $(\pi_2, f) : W \to X \times \mathbb{R}$ is proper since we have assumed that X is compact. For $n \gg d$ we get an embedding $W \subset X \times \mathbb{R} \times \mathbb{R}^{d-1+n}$ which lifts (π_2, f).

If $n \gg d$ the original embedding $W \subset X \times \mathbb{R} \times \mathbb{R}^{d-1+\infty}$ is isotopic to an embedding where the projection onto X is the submersion π and with (π, f) proper. (This is direct from [Phi67] when X is compact; and in general a slight extension.) We have constructed

$$\sigma \colon [X, \Omega^{\infty-1} MT(d)] \to D_d[X]. \tag{3.10}$$

Proposition 3.5. *The maps σ and ρ are inverse bijections.*

Proof. By construction $\rho \circ \sigma = \mathrm{id}$. The other composite $\sigma \circ \rho = \mathrm{id}$ uses that an element $W \in D_d(X)$ is concordant to one where W is replaced by $M \times \mathbb{R}$ and f by the projection; M is the inverse image of a regular value of f. The concordance is given in Lemma 2.5.2 of [MW02]. □

Remark 3.6. One can define σ also for non-compact X, but it requires a slight extension of [Phi67] to see that $(\pi_2, f) : W \to X \times \mathbb{R}$ can be taken to be proper. The proof of Theorem 3.4 above only uses (3.10) for compact X, in fact for X a sphere.

4. Proof of the main theorem

The proof uses an auxiliary sheaf of categories $D_d^\pitchfork$ and a zig-zag of functors

$$D_d \xleftarrow{\alpha} D_d^\pitchfork \xrightarrow{\gamma} C_d^\pitchfork \xleftarrow{\delta} C_d$$

The sheaf C_d is the cobordism category sheaf, defined in section 2.3 above, and $C_d^\pitchfork$ is the slightly larger sheaf, defined in the same section. The sheaf D_d is, by Theorem 3.4, a sheaf model of $\Omega^{\infty-1} MT(d)$. We regard D_d as a sheaf of categories with only identity morphisms. To prove the main theorem it will suffice to prove that α, γ and δ all induce weak equivalences.

Definition 4.1. Let $D_d^\pitchfork(X)$ denote the set of pairs (W, a) such that

(i) $W \in D_d(X)$,
(ii) $a : X \to \mathbb{R}$ is smooth,
(iii) $f : W \to \mathbb{R}$ is fiberwise transverse to a.

Thus, $D_d^\pitchfork$ is a subsheaf of $D_d \times \tilde{\mathbb{R}}$, where $\tilde{\mathbb{R}}$ is the representable sheaf $C^\infty(-, \mathbb{R})$. It is also a sheaf of posets, where $(W, a) \leq (W', a')$ when $W = W'$, $a \leq a'$ and $(a' - a)^{-1}(0) \subseteq X$ is open.

Recall from section 2.3 that $f : W \to \mathbb{R}$ is fiberwise transverse to $a : X \to \mathbb{R}$ if $f_x \colon W_x \to \mathbb{R}$ is transverse to $a(x) \in \mathbb{R}$ for all $x \in X$. By properness of (π, f), there will exist a smooth map $\varepsilon : X \to (0, \infty)$, such that the restriction of (π, f) to the open subset

$$W_\varepsilon = (\pi, f)^{-1}(X \times (a - \varepsilon, a + \varepsilon)),$$

is a (proper) submersion $W_\varepsilon \to X \times (a - \varepsilon, a + \varepsilon)$. Thus the class $[W_\varepsilon]$, as $\varepsilon \to 0$, is a well-defined element of $C_d^\pitchfork(X; a, a)$ and hence gives an object

$$\gamma(W, a) = ([W_\varepsilon], a, a) \in \operatorname{ob} C_d^\pitchfork(X).$$

This defines the functor $\gamma : D_d^\pitchfork \to C_d^\pitchfork$ on the level of objects, and it is defined similarly on morphisms.

Proposition 4.2. *The forgetful map $\alpha : \beta D_d^\pitchfork \to D_d$ is a weak equivalence.*

Proof. We apply the relative surjectivity criteria 2.5 to the map $\beta D_d^\pitchfork \to D_d$. The argument is completely analogous to the proof of Proposition 4.2.4 of [MW02].

First we show that $\beta D_d^\pitchfork(X) \to D_d(X)$ is surjective. Let $W \subseteq X \times \mathbb{R} \times \mathbb{R}^{d-1+\infty}$ be an element of $D_d(X)$. For each $x \in X$ we can choose $a_x \in \mathbb{R}$ such that a_x is a regular value of $f_x : W_x = \pi^{-1}(x) \to \mathbb{R}$. The same number a_x will be a regular value of $f_y : W_y \to \mathbb{R}$ for all y in a small neighborhood $U_x \subseteq X$ of x. Therefore we can pick a locally finite open covering $\mathcal{U} = (U_j)_{j \in J}$ of X, and real numbers a_j, so that $f_j \colon W_j \to \mathbb{R}$ is fiberwise transverse to a_j, where $W_j = W|U_j \in D_d(U_j)$. Thus (W_j, a_j) is an object of $D_d^\pitchfork(U_j)$ with $a_j \colon U_j \to \mathbb{R}$ the constant map.

For each finite subset $R \subseteq J$, set $W_R = W|U_R$ and $a_R = \min\{a_j \mid j \in R\}$. If $R \subseteq S$ then $a_S \leq a_R$ and (W_S, a_S, a_R) is an element $\varphi_{RS} \in N_1 D_d^\pitchfork(U_S)$. The pair $(\mathcal{U}, \Phi)$ with $\Phi = (\varphi_{RS})_{R \subseteq S}$ is an element of $\beta D_d^\pitchfork(U_S)$ that maps to W by α.

Second, let A be a closed subset of X, $W \subseteq X \times \mathbb{R} \times \mathbb{R}^{d-1+\infty}$ an element of $D_d(X)$, and suppose we are given a lift to $\beta D_d^\pitchfork(U')$ of the restriction of W to some open neighborhood U' of A. This lift is given by a locally finite open cover $\mathcal{U}' = \{U_j | j \in J\}$, together with smooth functions $a_R : U_R \to \mathbb{R}$, one for each finite non-empty $R \subseteq J$. Let $J' \subseteq J$ denote the set of j for which U_j is non-empty, and let $J'' = J - J'$.

Choose a smooth function $b : X \to [0, \infty)$ with $A \subseteq \operatorname{Int} b^{-1}(0)$ and $b^{-1}(0) \subseteq U'$. Let $q = 1/b : X \to (0, \infty]$. We can assume that $q(x) > a_R(x)$ for $R \subseteq J'$ (make U' smaller if not). For each $x \in X - U'$, we can choose an $a \in \mathbb{R}$ satisfying

(i) $a > q(x)$
(ii) a is a regular value for $f_x : \pi^{-1}(x) \to \mathbb{R}$.

The same number a will satisfy (i) and (ii) for all x in a small neighborhood $U_x \subseteq X - A$ of X, so we can pick an open covering $\mathcal{U}'' = \{U_j \mid j \in J''\}$ of $X - U'$, and real numbers a_j, such that (i) and (ii) are satisfied for all $x \in U_j$. The covering $\mathcal{U}''$ can be assumed locally finite. For each finite non-empty $R \subseteq J''$, set $a_R = \min\{a_j \mid j \in R\}$. For $R \subseteq J = J' \cup J''$, write $R = R' \cup R''$ with $R' \subseteq J'$ and $R'' \subseteq J''$, and define $a_R = a_{R'}$ if $R' \neq \emptyset$.

This defines smooth functions $a_R : U_R \to \mathbb{R}$ for all finite non-empty subsets $R \subseteq J$ (a_R is a constant function for $R \subseteq J''$) with the property that $R \subseteq S$ implies $a_S \le a_R|U_S$. This defines an element of $\beta D_d^{\pitchfork}(X)$ which lifts $W \in D_d(X)$ and extends the lift given near A. □

Proposition 4.3. *The inclusion functor $\gamma : D_d^{\pitchfork} \to C_d^{\pitchfork}$ induces an equivalence $B|D_d^{\pitchfork}| \to B|C_d^{\pitchfork}|$.*

Proof. We show that γ induces an equivalence $|N_k D_d^{\pitchfork}| \to |N_k C_d^{\pitchfork}|$ for all k, using the relative surjectivity criteria 2.5.

An element of $N_k C_d^{\pitchfork}(X)$ can be represented by a sequence of functions $a_0 \le \cdots \le a_k : X \to \mathbb{R}$, a function $\varepsilon : X \to (0, \infty)$, and a submanifold $W \subseteq X \times (a_0 - \varepsilon, a_k + \varepsilon) \times \mathbb{R}^{d-1+\infty}$. Choosing a diffeomorphism $X \times (a_0 - \varepsilon, a_k + \varepsilon) \to X \times \mathbb{R}$ which is the inclusion map on $X \times (a_0 - \varepsilon/2, a_k + \varepsilon/2)$, lifts the element to $N_k D_d^{\pitchfork}(X)$. This proves the absolute case and the relative case is similar. □

Proposition 4.4. *The forgetful functor $\delta : C_d \to C_d^{\pitchfork}$ induces a weak equivalence $B|C_d| \to B|C_d^{\pitchfork}|$.*

Proof. Again we prove the stronger statement that δ induces an equivalence $|N_k C_d| \to |N_k C_d^{\pitchfork}|$ for all k.

First, remember that two smooth maps $f\colon M \to P$ and $g\colon N \to P$ are called transversal if their product is transverse to the diagonal in $P \times P$. We apply Criteria 2.5, and first prove that δ is surjective on concordance classes. Let $\psi\colon \mathbb{R} \to [0,1]$ be a fixed smooth function which is 0 near $(-\infty, \frac{1}{3}]$ and is 1 near $[\frac{2}{3}, \infty)$, satisfying that $\psi' \ge 0$ and that $\psi' > 0$ on $\psi^{-1}((0,1))$.

Given smooth functions $a_0 \le a_1\colon X \to \mathbb{R}$ with $(a_1 - a_0)^{-1}(0) \subseteq X$ an open subset, we define $\varphi\colon X \times \mathbb{R} \to X \times \mathbb{R}$ by the formulas

$$\varphi(x,u) = (x, \varphi_x(u)),$$
$$\varphi_x(u) = \begin{cases} a_0(x) + (a_1(x) - a_0(x))\psi\big(\frac{u - a_0(x)}{a_1(x) - a_0(x)}\big) & \text{if } a_0(x) < a_1(x), \\ a_0(x) & \text{if } a_0(x) = a_1(x). \end{cases}$$

Suppose that $W \in C_d^{\pitchfork}(X; a_0, a_1)$ with $a_0 \le a_1$. The fiberwise transversality condition (iii) of Definition 2.6 implies that (π, f) and φ are transverse, and hence that

$$W_\varphi = \varphi^* W = \{(x, u, z) \mid \pi(z) = x, f(z) = \varphi_x(u)\}$$

is a submanifold of $X \times \mathbb{R} \times W$. Using the embedding $W \subset X \times \mathbb{R} \times \mathbb{R}^{d-1+\infty}$ we can rewrite W_φ as

$$W_\varphi = \{(x, u, r) \mid (x, \varphi_x(u), r) \in W\} \subseteq X \times \mathbb{R} \times \mathbb{R}^{d-1+\infty}.$$

It follows that

$$W_\varphi \cap \big(X \times (-\infty, a_0 + \varepsilon) \times \mathbb{R}^{d-1+\infty}\big) = M_0 \times (-\infty, a_0 + \varepsilon)$$
$$W_\varphi \cap \big(X \times (a_1 - \varepsilon, +\infty) \times \mathbb{R}^{d-1+\infty}\big) = M_1 \times (a_1 - \varepsilon, +\infty),$$

where $\varepsilon = 1$ on $(a_1 - a_0)^{-1}(0)$ and $\varepsilon = \frac{1}{3}(a_1 - a_0)$ otherwise. Thus W_φ defines an element of $C_d(X; a_0, a_1, \varepsilon)$, and in turn an element of $C_d(X; a_0, a_1)$.

We have left to check that W_φ is concordant to W in $C_d^\pitchfork(X; a_0, a_1)$. To this end we interpolate between the identity and our fixed function $\psi\colon \mathbb{R} \to [0,1]$. Define

$$\psi_s(u) = \rho(s)\psi(u) + (1 - \rho(s))u$$

with ρ any smooth function from $\mathbb{R}$ to $[0,1]$ for which $\rho = 0$ near $(-\infty, 0]$ and $\rho = 1$ near $[1, \infty)$. Define $\Phi\colon X \times \mathbb{R} \times \mathbb{R} \to X \times \mathbb{R}$ as $\Phi(x, s, u) = (x, \Phi_x(s, u))$ where

$$\Phi_x(s,u) = \begin{cases} a_0(x) + (a_1(x) - a_0(x))\psi_s\big(\frac{u - a_0(x)}{a_1(x) - a_0(x)}\big) & \text{if } a_0(x) < a_1(x), \\ \rho(s)a_0(x) + (1 - \rho(s))u & \text{if } a_0(x) = a_1(x). \end{cases}$$

Φ is transversal to (π, f), and the manifold

$$W_\Phi = \{((x,s), u, r) \mid (x, \Phi_x(s,u), r) \in W\} \subseteq (X \times \mathbb{R}) \times \mathbb{R} \times \mathbb{R}^{d-1+\infty}$$

defines the required concordance in $C_d^\pitchfork(X \times \mathbb{R})$ from W to W_φ.

We have proved that

$$\delta\colon N_0 C_d[X] \to N_0 C_d^\pitchfork[X] \quad \text{and}$$
$$\delta\colon N_1 C_d[X] \to N_1 C_d^\pitchfork[X]$$

are both surjective. The obvious relative argument is similar, and we can use Criteria 2.5. This proves that $\delta\colon |N_k C_d| \to |N_k C_d^\pitchfork|$ is a weak homotopy equivalence for $k = 0$ and $k = 1$. The case of general k is similar. □

Remark 4.5. There are versions of the sheaves $D_d^\pitchfork$, $C_d^\pitchfork$, C_d, where the functions $a : X \to \mathbb{R}$ are required to be *locally constant*. The proofs given in this section remain valid for these sheaves (the point is that in the proof of Proposition 4.2, we are choosing the functions $a_j : U_j \to \mathbb{R}$ locally constant anyway). This proves the claim in the last sentence of Remark 2.1(ii).

5. Tangential structures

We prove the version of the Main Theorem with tangential structures, as announced in the introduction. First we give the precise definitions.

Fix $d \geq 0$ as before, and let $BO(d) = G(d, \infty)$ denote the Grassmannian of d-planes in $\mathbb{R}^\infty$, $U_d \to BO(d)$ the universal d-dimensional vector bundle, and $EO(d)$ its frame bundle. Let

$$\theta : B \to BO(d)$$

be a Serre fibration (e.g. a fiber bundle). We think of θ as *structures* on d-dimensional vector bundles: If $f : X \to BO(d)$ classifies a vector bundle over X, then a θ-structure on the vector bundle is a map $l : X \to B$ with $\theta \circ l = f$.

An important class of examples comes from group representations. If G is a topological group and $\rho : G \to \mathrm{GL}(d, \mathbb{R})$ is a representation, then it induces a map $B\rho : BG \to B\,\mathrm{GL}(d, \mathbb{R}) \simeq BO(d)$, which we can replace by a Serre fibration.

In this case, a θ-structure is equivalent to a lifting of the structure group to G. These examples include $SO(d)$, $\mathrm{Spin}(d)$, $\mathrm{Pin}(d)$, $U(d/2)$ etc.

Another important class of examples comes from spaces with an action of $O(d)$. If Y is an $O(d)$-space, we let $B = EO(d) \times_{O(d)} Y$. If Y is a space with trivial $O(d)$-action, then a θ-structure amounts to a map from X to Y. If $Y = (O(d)/SO(d)) \times Z$, with trivial action on Z, then a θ-structure amounts to an orientation of the vector bundle together with a map from X to Z.

The proof of the main theorem applies almost verbatim if we add θ-structures to the tangent bundles of all d-manifolds in sight. We give the necessary definitions.

If $V \to X$ and $U \to Y$ are two vector bundles, a *bundle map* $V \to U$ is a continuous map of the total spaces of the vector bundles, which on each fiber of V restricts to a linear isomorphism onto a fiber of U. Let $\mathrm{Bun}(V, U)$ denote the space of all bundle maps, equipped with the compact-open topology. If $U = U_d$ is the universal bundle over $BO(d)$ (and X is a CW complex), then $\mathrm{Bun}(V, U)$ is contractible if V is d-dimensional.

A (non-identity) point in $\mathrm{mor}\, \mathcal{C}_d$ is given by (W, a_0, a_1), where $a_0 < a_1 \in \mathbb{R}$ and W is a submanifold (with boundary) of $[a_0, a_1] \times \mathbb{R}^{d-1+n}$, $n \gg 0$. The tangent spaces T_pW define a map

$$\tau_W : W \to G(d, n) \to BO(d),$$

covered by a bundle map $TW \to U_d$.

Definition 5.1. Let $\mathcal{C}_\theta$ be the category with morphisms (W, a_0, a_1, l), where $(W, a_0, a_1) \in \mathrm{mor}\, \mathcal{C}_d$ and $l : W \to B$ is a map satifying $\theta \circ l = \tau_W$. We topologize $\mathrm{mor}\, \mathcal{C}_\theta$ as in (2.4), but with $B_\infty(W)$ replaced with $B_\infty^\theta(W) = \mathrm{Emb}^\theta(W, [0, 1] \times \mathbb{R}^{d-1+\infty}) / \mathrm{Diff}(W)$, where Emb^θ is defined by the pullback square

$$\begin{array}{ccc} \mathrm{Emb}^\theta(W, [0,1] \times \mathbb{R}^{d-1+\infty}) & \longrightarrow & \mathrm{Bun}(TW, \theta^* U_d) \\ \downarrow & & \downarrow \theta \\ \mathrm{Emb}(W, [0,1] \times \mathbb{R}^{d-1+\infty}) & \xrightarrow{\tau_W} & \mathrm{Bun}(TW, U_d). \end{array} \tag{5.1}$$

The objects of $\mathcal{C}_\theta$ are topologized similarly.

The space $\mathrm{Bun}(TW, U_d)$ is contractible, so the inclusion of the fiber product in the product

$$\mathrm{Emb}^\theta(W, [0,1] \times \mathbb{R}^{d-1+\infty}) \to \mathrm{Emb}(W, [0,1] \times \mathbb{R}^{d-1+\infty}) \times \mathrm{Bun}(TW, \theta^* U_d)$$

is a homotopy equivalence. Dividing out the action of $\mathrm{Diff}(W)$ we get a homotopy equivalence

$$B_\infty^\theta(W) \xrightarrow{\simeq} E\,\mathrm{Diff}(W) \times_{\mathrm{Diff}(W)} \mathrm{Bun}(TW, \theta^* U_d).$$

Thus, up to homotopy,

$$\operatorname{ob}\mathcal{C}_\theta \simeq \coprod_M E\operatorname{Diff}(M) \times_{\operatorname{Diff}(M)} \operatorname{Bun}(\mathbb{R}\times TM, \theta^* U_d), \tag{5.2}$$

$$\operatorname{mor}\mathcal{C}_\theta \simeq \coprod_W E\operatorname{Diff}(W) \times_{\operatorname{Diff}(W)} \operatorname{Bun}(TW, \theta^* U_d), \tag{5.3}$$

where M runs over closed $(d-1)$-manifolds, one in each diffeomorphism class, and W runs over compact d-dimensional cobordisms, one in each diffeomorphism class. As before, $\operatorname{Diff}(W) \simeq \operatorname{Diff}(W, \{\partial_{\mathrm{in}}\}, \{\partial_{\mathrm{out}}\})$ denotes the topological group of diffeomorphisms that restrict to diffeomorphisms of the incoming and outgoing boundaries separately (or to product diffeomorphisms on a collar).

The left hand side of the homotopy equivalence (5.3) is the space of all morphisms in $\mathcal{C}_\theta$. The space of morphisms between two fixed objects can be determined similarly. We first treat the case $\theta = \mathrm{id}$. Let $c_0 = (M_0, a_0)$ and $c_1 = (M_1, a_1)$ be two objects of $\mathcal{C}_d$, given by real numbers $a_0 < a_1$, closed manifolds $M_\nu \subseteq \mathbb{R}^{d-1+\infty}$. Let W be a compact manifold and $h_0 : [0,1) \times M_0 \to W$ and $h_1 : (0,1] \times M_1 \to W$ be collars as in (2.3). Let

$$\operatorname{Emb}^\partial(W, [0,1]\times\mathbb{R}^{d-1+\infty}) \subseteq \operatorname{Emb}(W, [0,1]\times\mathbb{R}^{d-1+\infty})$$

be the subspace consisting of embeddings j which satisfy $j \circ h_0(t,x) = (t,x)$ for t sufficiently close to 0 and $j \circ h_1(t,x) = (t,x)$ for t sufficiently close to 1. Let $\operatorname{Diff}(W;\partial W) \subseteq \operatorname{Diff}(W)$ be the subgroup consisting of diffeomorphisms that restrict to the identity on a neighborhood of ∂W. This subgroup acts on $\operatorname{Emb}^\partial(W, [0,1]\times\mathbb{R}^{d-1+\infty})$ and we let $B^\partial_\infty(W)$ be the orbit space

$$B^\partial_\infty(W) = \operatorname{Emb}^\partial(W, [0,1]\times\mathbb{R}^{d-1+\infty})/\operatorname{Diff}(W;\partial W).$$

Then, up to homeomorphism, the space of morphisms is

$$\mathcal{C}_d(c_0, c_1) \cong \coprod_W B^\partial_\infty(W),$$

where the disjoint union is over cobordisms W from M_0 to M_1, one in each diffeomorphism class relative to M_0 and M_1. Since $\operatorname{Emb}^\partial(W, [0,1]\times\mathbb{R}^{d-1+\infty})$ is contractible, we get the homotopy equivalence

$$\mathcal{C}_d(c_0, c_1) \simeq \coprod_W B\operatorname{Diff}(W;\partial W).$$

The case of a general $\theta : B \to BO(d)$ is handled similarly. If $l_0 : M_0 \to B$ and $l_1 : M_1 \to B$ are two maps satisfying $\theta \circ l_\nu = \tau_{\mathbb{R}\times M_\nu}$ and $c_\nu = (M_\nu, a_\nu, l_\nu)$, $\nu = 0, 1$, then we get

$$\mathcal{C}_\theta(c_0, c_1) \simeq \coprod_W E\operatorname{Diff}(W;\partial W) \times_{\operatorname{Diff}(W;\partial W)} \operatorname{Bun}^\partial(TW, \theta^* U_d), \tag{5.4}$$

where $\operatorname{Bun}^\partial(TW, \theta^* U_d) \subseteq \operatorname{Bun}(TW, \theta^* U_d)$ is the subspace consisting of bundle maps which agree with the maps induced by l_0 and l_1 over a neighborhood of ∂W.

Let us consider the case of ordinary orientations in more detail. Here $B = BSO(d)$ is the oriented Grassmanian consisting of d-dimentional linear subspaces of $\mathbb{R}^\infty$ together with a choice of orientation, and $\theta : B \to BO(d)$ is the twofold covering space that forgets the orientation. Let W be a cobordism between the oriented manifolds M_0 and M_1. Then the set

$$\mathrm{Or}(W; \partial W) = \pi_0 \operatorname{Bun}^\partial(TW, \theta^* U_d)$$

is the set of orientations of W agreeing with the orientations given near ∂W (i.e. the collars h_0 and h_1 are oriented embeddings). Furthermore, the connected components of $\operatorname{Bun}^\partial(TW, \theta^* U_d)$ are contractible, so we get a homotopy equivalence

$$E\operatorname{Diff}(W; \partial W) \times_{\operatorname{Diff}(W;\partial W)} \operatorname{Bun}^\partial(TW, \theta^* U_d) \simeq \\ E\operatorname{Diff}(W; \partial W) \times_{\operatorname{Diff}(W;\partial W)} \mathrm{Or}(W; \partial W).$$

The stabilizer of an element of $\mathrm{Or}(W; \partial W)$ is the subgroup $\operatorname{Diff}^+(W; \partial W)$ of orientation preserving diffeomorphisms, restricting to the identity near the boundary. Thus we get

$$\mathcal{C}_d^+(c_0, c_1) \simeq \coprod_W B\operatorname{Diff}^+(W; \partial W),$$

where the disjoint union is over all oriented cobordisms W from M_0 to M_1, one in each oriented diffeomorphism class.

Definition 5.2. Let $\theta_{d,n} : B_{d,n} \to G(d,n)$ be the pullback

$$\begin{tikzcd} B_{d,n} \arrow[r] \arrow[d, "\theta_{d,n}"'] & B \arrow[d, "\theta"] \\ G(d,n) \arrow[r] & BO(d), \end{tikzcd}$$

and let $MT(\theta)$ be the spectrum whose $(n+d)$th space is $\mathrm{Th}(\theta_{d,n}^* U_{d,n})$.

The cofiber sequence (3.3) generalizes to a cofiber sequence

$$MT(\theta) \longrightarrow \Sigma^\infty B_+ \longrightarrow MT(\theta_{d-1}),$$

where θ_{d-1} is the pullback

$$\begin{tikzcd} B_{d-1} \arrow[r] \arrow[d, "\theta_{d-1}"'] & B \arrow[d, "\theta"] \\ BO(d-1) \arrow[r] & BO(d). \end{tikzcd}$$

With these definitions, the general form of the main theorem (as also stated in the introduction) is that for every tangential structure θ, there is a weak equivalence

$$B\mathcal{C}_\theta \simeq \Omega^{\infty-1} MT(\theta) = \operatorname*{colim}_{n\to\infty} \Omega^{d+n-1} \mathrm{Th}(\theta_{d,n}^* U_{d,n}^\perp).$$

The θ-versions of the sheaves used in section 4 to prove the special case $\theta = \mathrm{id}$, are defined as follows.

Definition 5.3. Let $W \in D_d(X)$. Let $T^\pi W$ be the fiberwise tangent bundle of the submersion $\pi : W \to X$. The embedding $W \subset X \times \mathbb{R}^{d+\infty}$ induces a canonical classifying map $T^\pi W : W \to BO(d)$. Let $D_\theta(X)$ be the set of pairs (W, l) with $W \in D_d(X)$ and $l : W \to B$ a map satisfying $\theta \circ f = T^\pi W$.

The sheaves C_d, $C_d^\pitchfork$ and $D_d^\pitchfork$ all consist of submanifolds $W \subseteq X \times \mathbb{R}^{d+n}$ such that the projection $\pi : W \to X$ is a submersion, together with some extra data. The tangential structure versions C_θ, $C_\theta^\pitchfork$ and $D_\theta^\pitchfork$ are defined in the obvious way: add a lifting $l : W \to B$ of the vertical tangent bundle $T^\pi W : W \to BO(d)$.

With these definitions, the proofs of section 4 apply almost verbatim. We note that the θ-versions of Theorem 3.4 and Proposition 4.4 use that θ is a Serre fibration.

6. Connectedness issues

This section, technically the hardest of the paper, compares the category $\mathcal{C}_\theta$ with the *positive boundary* subcategory $\mathcal{C}_{\theta,\partial}$. It is similar in spirit to section §6 of [MW02]. The two categories have the same space of objects. The space of morphisms of $\mathcal{C}_{\theta,\partial}$ is as in (2.4) and Definition 5.1, but taking only disjoint union over the W for which each connected component has non-empty *outgoing* boundary: if W is a cobordism from M_0 to M_1, then $\pi_0 M_1 \to \pi_0 W$ is surjective. In this section we prove

Theorem 6.1. *For $d \geq 2$ and any $\theta : B \to BO(d)$, the inclusion*

$$B\mathcal{C}_{\theta,\partial} \to B\mathcal{C}_\theta$$

is a weak equivalence.

In order to simplify the exposition we treat only the case $\theta = \mathrm{id}$. The general case of an arbitrary θ-structure is similar.

We say that a map $f : X \to Y$ of topological spaces is π_0-*surjective* if the induced map $\pi_0 X \to \pi_0 Y$ is surjective. The subsheaf $D_{d,\partial}^\pitchfork \subseteq D_d^\pitchfork$ is defined as follows: $(W, a_0, a_1) \in D_d^\pitchfork(\mathrm{pt})$ is in $D_{d,\partial}^\pitchfork(\mathrm{pt})$ if the inclusion

$$f^{-1}(a_1) \to f^{-1}[a_0, a_1]$$

is π_0-surjective. In general $\chi = (W, a_0, a_1) \in D_d^\pitchfork(X)$ is in $D_{d,\partial}^\pitchfork(X)$ if $\chi_{|\{x\}} \in D_{d,\partial}^\pitchfork(\{x\})$ for all $x \in X$. The proof given above that $|\beta D_d^\pitchfork| \simeq B\mathcal{C}_d$ (in Propositions 2.9, 4.3 and 4.4) is easily modified to show that $|\beta D_{d,\partial}^\pitchfork| \simeq B\mathcal{C}_{d,\partial}$. We will show that the composite map of sheaves $\beta D_{d,\partial}^\pitchfork \to \beta D_d^\pitchfork \to D_d$ satisfies the relative lifting criteria 2.5 for all $d \geq 2$.

6.1. **Discussion.** We describe the ideas involved and indicate the issues in proving that the map $\beta D^{\pitchfork}_{d,\partial} \to D_d$ is a weak equivalence.

As a first approximation we can try to repeat the proof for $\beta D^{\pitchfork}_d \to D_d$ (in Proposition 4.2), by choosing regular values $a_x \in \mathbb{R}$ for $f_x : W_x \to \mathbb{R}$ "at random" (using Sard's theorem), and using that a_x is a regular value for $f_y : W_y \to \mathbb{R}$ also for y in a small neighborhood U_x of $x \in X$. This will produce an element $(W, (U_j, a_j)_{j\in J}) \in \beta D^{\pitchfork}_d(X)$ but in general there is, of course, no reason to expect to get an element of $\beta D^{\pitchfork}_{d,\partial}(X) \subseteq \beta D^{\pitchfork}_d(X)$. The idea is now to deform (i.e. change by a concordance) the underlying $W \in D_d(X)$ to an element $W' \in D_d(X)$ such that W' together with the regular values a_j (possibly slightly perturbed) defines an element of $\beta D^{\pitchfork}_{d,\partial}(X)$.

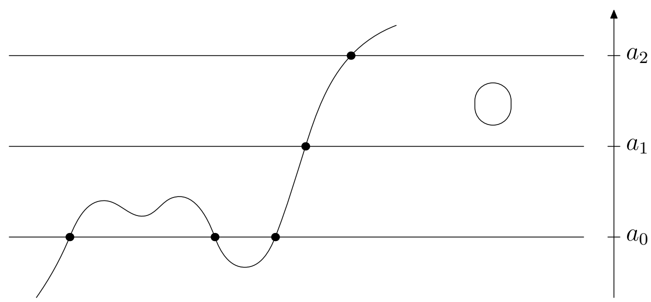

FIGURE 1.

It is instructive to first consider the case $X = \mathrm{pt}$. Given an element $(W, a_0 < \cdots < a_k) \in N_k D^{\pitchfork}_d(\mathrm{pt})$, it is easy to see that there is a concordance $H \in D_d(\mathbb{R})$ from W to W' such that $(W', a_0 < \cdots < a_k) \in N_k D^{\pitchfork}_{d,\partial}(\mathrm{pt})$. Roughly, we have to get rid of some *local maxima*, with values between a_0 and a_k, of the function $f : W \to \mathbb{R}$ cf. Figure 1. A naive way to do that is to "pull them up", i.e. if $p \in W$ is near a "local maximum" for $f : W \to \mathbb{R}$, then we can change f near p to have $f(p) > a_k$ cf. Figure 2. A better way (for reasons explained below) to get rid of a local maximum, is given in Lemma 6.2 below.

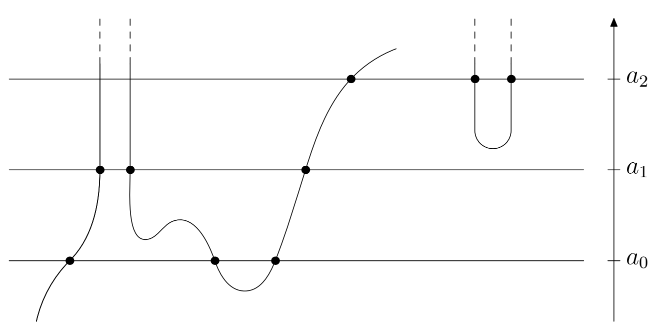

FIGURE 2.

For general X it is equally easy to solve the problem *locally*. Given $W \in D_d(X)$, suppose we have chosen regular values $a_j \in \mathbb{R}$ and corresponding open covering

$U_j \subseteq X$, $j \in J$, such that $(W, (a_j, U_j)_{j\in J})$ defines an element of $\beta D_d^{\pitchfork}(X)$. Given $x \in X$ it is easy (as in the case $X = \mathrm{pt}$) to find a small neighborhood $U_x \subseteq X$ and a concordance $H_x \in D_d(U_x \times \mathbb{R})$ from $W|U_x$ to $W' \in D_d(U_x)$ such that $(W', (a_j, U_j \cap U_x)_{j\in J})$ defines an element of $\beta D_{d,\partial}^{\pitchfork}(U_x)$. We now need to *glue* these local constructions.

The locally defined concordance $H_x \in D_d(U_x \times \mathbb{R})$ can be assumed to extend to $H_x \in D_d(X \times \mathbb{R})$. Namely we may choose a bump function $\lambda : X \to [0,1]$, supported in U_x, and which is 1 in a smaller neighborhood $U'_x \subseteq U_x$, and let $h : U_x \times \mathbb{R} \to U_x \times \mathbb{R}$ be given by $h(x,t) = (x, t\lambda(x))$. Then $H'_x = h^* H_x \in D_d(U_x \times \mathbb{R})$ is a concordance which is constant outside the support of λ, so it extends to a concordance $H'_x \in D_d(X \times \mathbb{R})$. Moreover $H'_x|(U'_x \times \mathbb{R}) = H_x|(U'_x \times \mathbb{R})$. Thus H'_x is a concordance from W to $W' \in D_d(X)$, such that $W'|U'_x \in D_d(U'_x)$ lifts to $\beta D_{d,\partial}(U'_x)$. Also W and W' agree outside $U_x \supseteq U'_x$.

We have described how, given a way of getting rid of a single local maxima, to deform an element $W \in D_d(X)$ into an element $W' \in D_d(X)$, with the property that $W'|U'_x$ lifts to $\beta D_{d,\partial}^{\pitchfork}(U'_x)$, and such that W and W' agree outside a larger open neighborhood $U_x \supseteq U'_x$. Roughly, the idea is now to apply such a construction for sufficiently many $x \in X$, enough that the sets U'_x cover X. For this to work there is one critical issue, however. Namely it is essential that the local construction used to get rid of fiberwise local maxima over U'_x does not create new fiberwise local maxima over $U_x - U'_x$. Without this, the idea to "apply such a construction for sufficiently many $x \in X$" will not work.

The naive idea of "pulling local maxima up" will not work, precisely for this reason. If we "pull up" a fiberwise local maximum over U'_x, we have to pull less and less over $U_x - U'_x$ (as specified by the bump function λ), which will give rise to fiberwise local maxima of $f' : W' \to \mathbb{R}$ over $U_x - U'_x$ which are not fiberwise local maxima of $f : W \to \mathbb{R}$.

Thus we will need a way of deforming $f : W \to \mathbb{R}$ to get rid of local maxima without creating new ones in the process. Such a construction is described in Lemma 6.2 below. It describes a family of maps $f_t : K_t \to \mathbb{R}$, $t \in [0,1]$ from d-manifolds K_t, such that f_0 is the constant map $0 : \mathbb{R}^d \to \mathbb{R}$, such that $f_1 : \mathbb{R}^d - \{0\} \to \mathbb{R}$ has $\lim_{x\to 0} f(x) = +\infty$, and such that $f_t : K_t \to \mathbb{R}$ has no local maxima, except some with value $0 \in \mathbb{R}$, for any $t \in [0,1]$. Moreover each K_t contains the open subset $\mathbb{R}^d - D^d \subseteq K_t$ and $f_t|(\mathbb{R}^d - D^d) = 0$.

6.2. **Surgery.** The geometric construction is based on the following lemma. Let us say that a map $f : M \to N$ is proper *relative to* an open set $U \subseteq M$, if $f_{|M-U} : M - U \to N$ is proper.

Lemma 6.2. *There exists a smooth $(d+1)$-manifold K containing $U = \mathbb{R} \times (\mathbb{R}^d - D^d)$ as an open subset, and smooth maps $(\pi, f) : K \to \mathbb{R} \times \mathbb{R}$, such that*

(i) π is a submersion, and (π, f) is proper relative to U. In particular, if we let $K_t = \pi^{-1}(t)$ and $U_t = U \cap K_t = \{t\} \times (\mathbb{R}^d - D^d)$, then $f_t : K_t \to \mathbb{R}$ is proper relative to U_t.

(ii) $(\pi, f)(t, x) = (t, 0)$ *for all* $(t, x) \in U \subseteq K$.
(iii) $K_0 = \{0\} \times \mathbb{R}^d$ *and* $f_0 : K_0 \to \mathbb{R}$ *is the zero function.*
(iv) *For all* $t \in [0, 1]$ *and all* $a_0 < a_1 \in \mathbb{R}$, *the following inclusions are* π_0-*surjections*

$$\begin{aligned} U_t \amalg f_t^{-1}(a_1) \to f_t^{-1}([a_0, a_1]) \quad &\text{if } 0 \in [a_0, a_1] \\ f_t^{-1}(a_1) \to f_t^{-1}([a_0, a_1]) \quad &\text{if } 0 \notin [a_0, a_1]. \end{aligned}$$

(v) *For all* $a_0 < a_1 \in \mathbb{R}$, *the inclusion*

$$f_1^{-1}(a_1) \to f_1^{-1}([a_0, a_1])$$

is a π_0-*surjection.*
(vi) $K_1 = \{1\} \times (\mathbb{R}^d - \{0\})$ *and* $f_1 : K_1 \to \mathbb{R}$ *is non-negative and has* $0 \in \mathbb{R}$ *as only critical value.*
(vii) $T^\pi K$ *is a trivial vector bundle.*

The last property, that $T^\pi K$ be a trivial vectorbundle, is needed to make the constructions work in the presence of θ-structures.

As stated, the lemma is true also for $d = 1$, but is useful only for $d > 1$. For $d > 1$ the set U_t is connected, and the properties (iii) and (iv) say that the number of elements in the quotient

$$Q_t = \pi_0(f_t^{-1}[a_0, a_1])/\pi_0(f_t^{-1}(a_1))$$

is never larger than the number of elements in Q_0. For $0 \in [a_0, a_1]$ and $d = 1$, the inclusion $U_t \to f_t^{-1}([a_0, a_1])$ defines an element $[U_t] \in Q_t$, and (v) says that $[U_0] \in Q_0$ is not the basepoint, then Q_1 is strictly smaller than Q_0.

Proof. We will construct K as a certain pullback of a 2-manifold L which we first construct. L will come with an immersion $(\pi, j) : L \to [0, 4] \times [0, \infty)$ and a function $f : L \to \mathbb{R}$. L will be glued from four pieces $L^1, \dots, L^4$ which we construct individually. The pieces L^1, L^2 and L^4 will be subsets of $[0, 1] \times [0, \infty)$, and L^3 will be the disjoint union of three open subsets of $[0, 1] \times [0, \infty)$. In all cases, $(\pi, j) : L^\nu \to [0, 1] \times [0, \infty)$ will be given by the inclusions.

Let $\rho : [0, \infty) \to [0, 1]$ be a smooth function with $\mathrm{supp}(\rho) = [0, 1]$, $\rho(0) = 1$, and $\rho'(r) \leq 0$. For $s \in [0, 1]$ let $q_s(r) = \rho(r^2)\frac{1-s}{r^2+s}$ and let g_s and $\hat{g}_s$ be the functions given by

$$\begin{aligned} g_s(r) &= -q_s(r) - q_s(r-2) + q_0(r-1) \\ \hat{g}_s(r) &= \mathrm{sgn}(r(r-2))\left(-q_0(r) - q_0(r-2) + q_{1-s}(r-1) - \frac{1-s}{s}\right) + \frac{1-s}{s}. \end{aligned}$$

$g_s(r)$ is defined unless $r = 1$ or $(s, r) \in \{0\} \times \{0, 2\}$. $\hat{g}_s(r)$ is defined unless $r \in \{0, 2\}$ or $(s, r) = (1, 1)$ or $(s, r) \in \{0\} \times [0, 2]$. It is easily checked that $g_s'(r) = 0$ only if $r \geq 3$, if $(s, r) \in (0, 1] \times \{0, 2\}$, or if $(s, r) \in \{1\} \times [2, \infty)$. Similarly $\hat{g}_s'(r) = 0$ only if $r \geq 3$ or $(s, r) \in (0, 1) \times \{1\}$. All isolated critical points of g_s and $\hat{g}_s$ are local minima.

Define functions $f^\nu : L^\nu \to \mathbb{R}$ for $\nu = 1, 2, 4$ by the following formulas, using the (calculus) convention that the set $L^\nu \subseteq [0,1] \times [0,\infty)$ is the largest open set for which the definitions make sense.

$$\begin{aligned} f^1(t,r) &= \hat{g}_0(r + 3(1-t)) \\ f^2(t,r) &= \hat{g}_t(r) \\ f^4(t,r) &= g_t(r+t). \end{aligned}$$

To define f^3, let $L^3 = L^3_- \amalg L^3_+ \amalg L^3_0$, where

$$\begin{aligned} L^3_- &= \{(t,r) \in [0,1] \times [0,\infty) \mid t < r < t+1\} \\ L^3_+ &= \{(t,r) \in [0,1] \times [0,\infty) \mid (1-t) < r < (2-t)\} \\ L^3_0 &= [0,1] \times (2,\infty). \end{aligned}$$

Let $f^3 = f^3_- \amalg f^3_+ \amalg f^3_0$, where

$$f^3_\varepsilon(t,r) = \hat{g}_1(r + \varepsilon t).$$

It is easily checked that $f^1(1,r) = f^2(0,r)$, $f^2(1,r) = f^3(0,r)$ and $f^3(1,r) = f^4(0,r)$, so they glue to a continuous function $\tilde{f} : \tilde{L} \to \mathbb{R}$, where $\tilde{L}$ is glued from $L^1, \dots, L^4$. $\tilde{L}$ is a smooth manifold and comes with an immersion $(\tilde{\pi}, \tilde{j}) : \tilde{L} \to [0,4] \times [0,\infty)$. The 2-manifold $\tilde{L}$ is sketched in Figure 3, which also depicts the map $\tilde{\pi} : \tilde{L} \to [0,4]$ as the projection onto the horizontal axis and $\tilde{j} : \tilde{L} \to [0,\infty)$ as the projection onto the vertical axis.

The function $\tilde{f}$ is not smooth in the t-variable along the gluing lines. To fix that, we choose a function $\sigma : [0,4] \to [0,4]$ which for each $n = 1, 2, 3$ has $\sigma(t) = n$ for all t near n. Then let L be defined by the pullback diagram

$$\begin{tikzcd} L \arrow[r, "\overline{\sigma}"] \arrow[d, "\pi"'] & \tilde{L} \arrow[d, "\tilde{\pi}"] \\ {[0,4]} \arrow[r, "\sigma"] & {[0,4]}, \end{tikzcd}$$

and let $j = \tilde{j} \circ \overline{\sigma} : L \to [0,\infty)$ and $f = \tilde{f} \circ \overline{\sigma} : L \to \mathbb{R}$. The resulting $f : L \to \mathbb{R}$ is then smooth.

Let $\lambda : \mathbb{R} \to [0,1]$ be a smooth function which is 0 near $(-\infty, 0]$ and 1 near $[1,\infty)$ and has $\lambda' > 0$ on $\lambda^{-1}((0,1))$. Let $g : \mathbb{R} \times \mathbb{R}^d \to [0,4] \times [0,\infty)$ be the map given by

$$g(t,x) = (4\lambda(t), 3|x|^2).$$

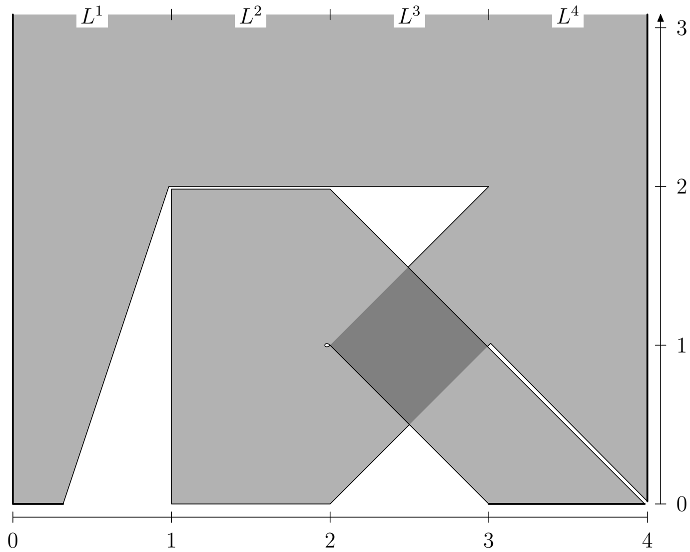

FIGURE 3. Image of $(\tilde{\pi}, \tilde{j}) : \tilde{L} \to [0,4] \times [0,\infty)$.

To construct the map $(\pi, f) : K \to \mathbb{R} \times \mathbb{R}$ of the proposition, define K as the pullback in the diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d, "(\pi,j)"'] & L \arrow[r, "f"] \arrow[d, "(\pi,j)"] & \mathbb{R} \\ \mathbb{R} \times \mathbb{R}^d \arrow[r, "g"] & {[0,4] \times [0,\infty).} & \end{tikzcd} \tag{6.1}$$

Then $(\pi, j) : K \to \mathbb{R} \times \mathbb{R}^d$ is a codimension 0 immersion, and over $U = \mathbb{R} \times (\mathbb{R}^d - D^d)$ it is a diffeomorphism. The diagram also provides a map $f : K \to \mathbb{R}$, and it is easily seen that $(\pi, f) : K \to \mathbb{R} \times \mathbb{R}$ satisfies the first six properties of the proposition. The differential of $(\pi, j) : K \to \mathbb{R} \times \mathbb{R}^d$ defines a trivialization of the d-dimensional vector bundle $T^\pi K$. $\square$

The manifold K and the map $(\pi, f) : K \to \mathbb{R}^d \times \mathbb{R}$ are illustrated in Figure 4, which shows the d-manifold $K_t = \pi^{-1}(t)$ for $d = 1$ and various values of $t \in [0,1]$. The horizontal axis is $[-1,1] = D^d \subseteq \mathbb{R}^d$ and the projection is the immersion $j_t : K_t \to \mathbb{R}^d$. The vertical axis is $(-\infty, \infty)$ and the projection is the function $f_t : K_t \to (-\infty, \infty)$. The small arrows indicate how K_t changes when t increases.

Given an element $W \in D_d(\mathrm{pt})$, assume $e : \mathbb{R}^d \to W$ is an embedding with $e(\mathbb{R}^d) \subseteq f^{-1}(r)$ for some $r \in \mathbb{R}$. Then $W \times \mathbb{R} \in D_d(\mathbb{R})$ has an embedded $\mathbb{R}^d \times \mathbb{R}$ from which we can remove $D^d \times \mathbb{R}$ and glue in the manifold K from the above Lemma 6.2 along the embedded $(\mathbb{R}^d - D^d) \times \mathbb{R}$. This gluing is over $\mathbb{R}$ if we equip K with the map $f + r : K \to \mathbb{R}$ and we get a concordance $W^e \in D_d(\mathbb{R})$ starting at $W \in D_d(\{0\})$. We will describe an enhanced version of this construction where we start with $W \in D_d(X)$ and a finite set of embeddings $e_\tau : X \times \mathbb{R}^d \to W$ ($\tau \in T$) such that $r_\tau(x) = f \circ e_\tau(x, u)$ is independent of $u \in \mathbb{R}^d$. The enhanced

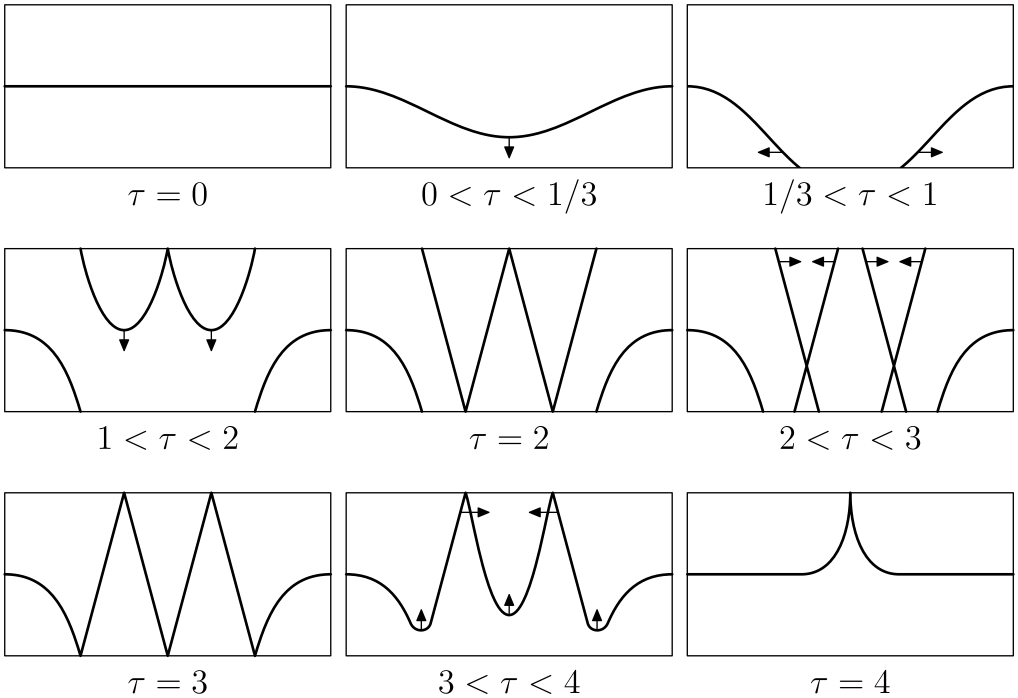

FIGURE 4. $(f_t, j_t)(K_t)$ for $d = 1$ and various values of $\tau = \sigma(4\lambda(t)) \in [0, 4]$.

construction will give an element $W^e \in D_d(X \times \mathbb{R}^T)$ which upon restriction to $X \times \{1\}^T$ is an element where the "local maxima" at $e_\tau(x, 0)$ have disappeared.

Definition 6.3. Let X be a manifold and T a finite set. Let $r : X \times T \to \mathbb{R}$ be smooth. For $\tau \in T$, let $q_{\tau,r} : (X \times \mathbb{R}^T) \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be the map

$$q_{\tau,r}((x, l), t) = (l_\tau, t - r(x, \tau)), \quad l = (l_\tau)_{\tau \in T} \ .$$

Considering K as a space over $\mathbb{R} \times \mathbb{R}$ via the map (π, f) from (6.1), we get a manifold $q^*_{\tau,r}K$ over $(X \times \mathbb{R}^T) \times \mathbb{R}$, containing $q^*_{\tau,r}U = (X \times \mathbb{R}^T) \times (\mathbb{R}^d - D^d)$ as an open subset. Let

$$K^r = \coprod_{\tau \in T} q^*_{\tau,r} K, \qquad U^r = \coprod_{\tau \in T} q^*_{\tau,r} U \subseteq K^r.$$

This comes equipped with a map $(\pi^r, f^r) : K^r \to (X \times \mathbb{R}^T) \times \mathbb{R}$ which is proper relative to $U^r = (X \times \mathbb{R}^T) \times \coprod_T (\mathbb{R}^d - D^d)$, and $\pi^r : K^r \to X \times \mathbb{R}^T$ is a submersion.

Remark 6.4. This behaves well under union in the T-variable. If $T = T_0 \amalg T_1$ and $r_\nu : X \times T_\nu \to \mathbb{R}$, $\nu = 0, 1$ are the restrictions of r, then

$$K^r = \mathrm{proj}^*_{X \times \mathbb{R}^{T_0}}(K^{r_1}) \amalg \mathrm{proj}^*_{X \times \mathbb{R}^{T_1}}(K^{r_0})$$

where the indicated projections are $X \times \mathbb{R}^T \to X \times \mathbb{R}^{T_\nu}$, $\nu = 0, 1$.

Construction 6.5. *Let $W \in D_d(X)$, and let T be a finite set. Let $r : X \times T \to \mathbb{R}$ be smooth. Then $X \times \coprod_T \mathbb{R}^d = X \times T \times \mathbb{R}^d$ is a space over $X \times \mathbb{R}$ via the projection composed with r. Let*

$$e : X \times \coprod_T \mathbb{R}^d \to W$$

be an embedding over $X \times \mathbb{R}$, i.e. with $\pi \circ e(x,\tau,u) = x$ and $f \circ e(x,\tau,u) = r(x,\tau)$. This induces an embedding

$$\tilde{e} : (X \times \mathbb{R}^T) \times \coprod_T \mathbb{R}^d \to \operatorname{proj}_X^* W,$$

where $\operatorname{proj}_X : X \times \mathbb{R}^T \to X$ is the projection. Let W^e be the pushout

$$\begin{tikzcd} U^r \arrow[r, "\tilde{e}"] \arrow[d] & \operatorname{proj}_X^* W - \tilde{e}(X \times \mathbb{R}^T \times \coprod_T D^d) \arrow[d] \\ K^r \arrow[r] & W^e \end{tikzcd} \tag{6.2}$$

This gives a manifold W^e over $(X \times \mathbb{R}^T) \times \mathbb{R}$ which defines an element of $D_d(X \times \mathbb{R}^T)$.

Elements of $D_d(X \times \mathbb{R}^T)$ are submanifolds of $(X \times \mathbb{R}^T) \times \mathbb{R} \times \mathbb{R}^{d-1+\infty}$, so strictly speaking the construction of W^e includes a choice of an embedding

$$\varphi : W^e \to (X \times \mathbb{R}^T) \times \mathbb{R} \times \mathbb{R}^{d-1+\infty}$$

extending the given map $W^e \to (X \times \mathbb{R}^T) \times \mathbb{R}$. Then the image $\varphi(W^e)$ is an element of $D_d(X \times \mathbb{R}^T)$. The element $\operatorname{proj}_X^* W \in D_d(X \times \mathbb{R}^T)$ has a preferred embedding $i : \operatorname{proj}_X^* W \to (X \times \mathbb{R}^T) \times \mathbb{R} \times \mathbb{R}^{d-1+\infty}$ (namely the inclusion), and it is convenient to assume that φ and i agree on the subspace $\operatorname{proj}_X^* W - \tilde{e}(X \times \mathbb{R}^T \times \coprod_T D^d)$. Such an embedding φ can always be chosen, and is unique up to isotopy. It is irrelevant for the arguments which φ we choose, and therefore we omit it from the notation, writing $W^e \in D_d(X \times \mathbb{R}^T)$ instead of $\varphi(W^e)$.

6.3. **Connectivity.** We will apply the surgery construction of the previous section to a morphism $(W, a_0, a_1) \in D_d^{\pitchfork}(X)$ with $a_0 < a_1$. The resulting $W^e \in D_d(X \times \mathbb{R}^T)$ will usually *not* give rise to an element $(W^e, a_0, a_1) \in D_d^{\pitchfork}(X \times \mathbb{R}^T)$ because $f^e : W^e \to \mathbb{R}$ might not be fiberwise transverse to a_0, a_1. Let $V = V(a_0, a_1) \subseteq X \times \mathbb{R}^T$ be the open set of points (x, l) for which $f^e_{(x,l)} : W^e_{(x,l)} \to \mathbb{R}$ is transverse to $a_0(x)$ and $a_1(x)$. Then we have $(W^e, a_0, a_1)_{|V} \in D_d^{\pitchfork}(V)$. By Sard's theorem, any $(x, t) \in X \times \mathbb{R}^T$ is in $V(b_0, b_1)$ for some b_0, b_1 arbitrarily close to a_0, a_1. The goal is to use these concordances to get an element of $D_{d,\partial}^{\pitchfork}$. Since the condition for being in $D_{d,\partial}^{\pitchfork} \subseteq D_d^{\pitchfork}$ is pointwise, we restrict attention to the case $X = \mathrm{pt}$ in the following propositions.

Proposition 6.6. *Let $(W, a_0, a_1) \in D_d^{\pitchfork}(\mathrm{pt})$ with $a_0 < a_1$. Let $r : T \to \mathbb{R}$ and $e : \coprod_T \mathbb{R}^d \to W$ be as in Construction 6.5. Let $V = V(a_0, a_1) \subseteq \mathbb{R}^T$ be as above.*

(i) If $r(\tau) \neq a_0, a_1$ for all $\tau \in T$, then $\{0,1\}^T \subseteq V$.
(ii) If $(W, a_0, a_1) \in D_{d,\partial}^{\pitchfork}(\mathrm{pt})$, then $(W^e, a_0, a_1)_{|V} \in D_{d,\partial}^{\pitchfork}(V)$.
(iii) If $(W, a_0, a_1) \in D_d^{\pitchfork}(\mathrm{pt})$, $a_0 < r < a_1$, and if

$$f^{-1}(a_1) \amalg \coprod_T \mathbb{R}^d \to f^{-1}([a_0, a_1])$$

is π_0-surjective, then the restriction to $\{1\}^T \subseteq \mathbb{R}^T$ defines an element $(W, a_0, a_1)_{\{1\}^T} \in D_{d,\partial}^{\pitchfork}(\{1\}^T)$.

Proof. Let $l \in \{0,1\}^T$. By Lemma 6.2(vi) we get that critical values of $f_l : W_l^e \to \mathbb{R}$ will be either critical values of $f : W \to \mathbb{R}$, or values $r(\tau)$ for $\tau \in T$ with $l_\tau = 1$. This proves *(i)*. *(ii)* follows from Lemma 6.2*(iv)* and *(iii)* follows in the same way from Lemma 6.2*(v)*. □

If not $l \in V(a_0, a_1)$, then $l \in V(b_0, b_1)$ for some b_0, b_1 near a_0, a_1. We have the following corollary of the above proposition.

Corollary 6.7. *Let $(W, a_0, a_1) \in D_d^{\pitchfork}(\mathrm{pt})$. Let U_0 and U_1 be small open intervals in $\mathbb{R}$ around a_0 and a_1, respectively, consisting of regular values of f. Let $r : T \to \mathbb{R}$ and $e : \coprod_T \mathbb{R}^d \to W$ be as in Construction 6.5. Let $T = T_0 \amalg T_1$ and assume $\sup U_0 < r(\tau) < \inf U_1$ for $\tau \in T_1$, and that*

$$f^{-1}(a_1) \amalg \coprod_{T_1} \mathbb{R}^d \to f^{-1}([a_0, a_1])$$

is π_0-surjective. Then

$$(W_l^e, b_0, b_1) \in D_{d,\partial}^{\pitchfork}(\{l\})$$

for all $b_0, b_1 \in U_0 \cup U_1$ with $b_0 < b_1$, and all $l \in V(b_0, b_1) \cap \left(\mathbb{R}^{T_0} \times \{1\}^{T_1}\right)$.

Proof. If $b_0 \in U_0$ and $b_1 \in U_1$ then, since U_0 and U_1 are connected and consist of regular values of f,

$$f^{-1}(b_1) \amalg \coprod_{T_1} \mathbb{R}^d \to f^{-1}([b_0, b_1]) \tag{6.3}$$

will also be π_0-surjective. If $b_0, b_1 \in U_1$ or if $b_0, b_1 \in U_1$, then $[b_0, b_1]$ consists of regular values of f, so $f^{-1}([b_0, b_1]) \cong f^{-1}(b_1) \times [b_0, b_1]$, so the inclusion (6.3) is π_0-surjective in this case too. Therefore, by Proposition 6.6(iii) the element $W^{e_1} \in D_d(\mathbb{R}^{T_1})$ will have

$$(W^{e_1}_{\{1\}^{T_1}}, b_0, b_1) \in D_{d,\partial}^{\pitchfork}(\{1\}^{T_1}).$$

It follows from Remark 6.4 that the construction of $W^e \in D_d(X \times \mathbb{R}^T)$ enjoys the following naturality property. If $T = T_0 \amalg T_1$, then we can restrict e to $e_\nu : X \times \coprod_{T_\nu} \mathbb{R}^d \to W$, $\nu = 0, 1$. By construction (diagram (6.2)), the element

W^{e_1} contains the open subset $\mathrm{proj}_X^* W - \tilde{e}_1(X \times \mathbb{R}^{T_1} \times \coprod_{T_1} D^d)$ and hence e_0 defines an embedding

$$\mathrm{proj}_X^*(e_0) : (X \times \mathbb{R}^{T_1}) \times \coprod_{T_0} \mathbb{R}^d \to W^{e_1}.$$

The naturality property is that

$$(W^{e_1})^{\mathrm{proj}_X^*(e_0)} = W^e.$$

Restricting to $\{1\}^{T_1} \times \mathbb{R}^{T_0}$ we have

$$W^e_{\{1\}^{T_1} \times \mathbb{R}^{T_0}} = (W^{e_1}_{\{1\}^{T_1}})^{\mathrm{proj}_X^*(e_0)}.$$

The claim now follows from Proposition 6.6(ii) above. $\square$

We will say that an open set $U_0 \subseteq X \times \mathbb{R}$ is a *tube* around a_0 if it contains the graph of a_0, and if the intersection $U_0 \cap \{x\} \times \mathbb{R}$ is an interval consisting of regular values of $f_x : W_x \to \mathbb{R}$ for all $x \in X$.

Definition 6.8. For a function $\lambda : X \times T \to [0,1]$, let $\hat{\lambda} : X \times \mathbb{R} \to X \times \mathbb{R}^T$ denote the adjoint $\hat{\lambda}(x,t) = (x, t\lambda(x))$. Given $r : X \times T \to \mathbb{R}$ and $e : X \times \coprod_T \mathbb{R}^d \to W$ as in Construction 6.5, let $W^{e,\lambda} \in D_d(X \times \mathbb{R})$ denote the pullback of W^e along $\hat{\lambda}$.

If $T = T_0 \amalg T'$ and $\lambda_{|X \times T_0} = 0$, then $W^{e,\lambda} = W^{e',\lambda'}$, where e' and λ' are the restrictions to $T' \subseteq T$. The following corollary follows immediately from Corollary 6.7 above.

Corollary 6.9. *Let $(W, a_0, a_1) \in D_d^{\pitchfork}(X)$. Let r, e, λ be as in Definition 6.8. Let $W^{e,\lambda} \in D_d(X \times \mathbb{R})$ be the resulting element. Let U_0, U_1 be tubes around a_0 and a_1. Assume that there is a subset $T_1 \subseteq T$ with $\lambda_{|X \times T_1} = 1$, such that the graph of $r_{|X \times T_1}$ is above U_0 and below U_1, and such that*

$$f_x^{-1}(a_1(x)) \amalg \coprod_{T_1} \mathbb{R}^d \to f_x^{-1}([a_0(x), a_1(x)])$$

is π_0-surjective for all x.

For all $b_0, b_1 : X \to \mathbb{R}$ with $b_0 < b_1$ and $\mathrm{graph}(b_\nu) \subseteq U_0 \cup U_1$*, let $\hat{V}(b_0, b_1)$ denote the intersection $X \times \{1\} \cap \hat{\lambda}^{-1} V(b_0, b_1)$. Then the resulting element*

$$(W^{e,\lambda}, b_0, b_1)_{|\hat{\lambda}^{-1} V(b_0,b_1)} \in D_d^{\pitchfork}(\hat{\lambda}^{-1} V(b_0, b_1))$$

restricts to an element

$$(W^{e,\lambda}, b_0, b_1)_{|\hat{V}(b_0,b_1)} \in D_{d,\partial}^{\pitchfork}(\hat{V}(b_0, b_1))$$

Thus, we get a *concordance* from $W = W^{e,\lambda}_{|X \times \{0\}} \in D_d(X \times \{0\})$ to the element $W^{e,\lambda}_{|X \times \{1\}} \in D_d(X \times \{1\})$ and the latter element lifts over $\hat{V}(b_0, b_1)$ to morphisms in $D_{d,\partial}^{\pitchfork}$.

6.4. **Parametrized surgery.** So far we have described how to perform surgery on $W \in D_d(X)$ along an embedding $e : X \times \coprod_T \mathbb{R}^d \to W$. If we only have such embeddings given *locally* in X, then we can perform the surgeries locally and glue them together using appropriate partitions of unity. More precisely we have the following construction.

Construction 6.10. *Let $(p, r) : E \to X \times \mathbb{R}$ be smooth, with $p : E \to X$ etale (local diffeomorphism). Let $e : E \times \mathbb{R}^d \to W$ an embedding over $X \times \mathbb{R}$. Let $\lambda : E \to [0, 1]$ be a smooth map with $p|\operatorname{supp}\lambda$ proper. Define an element $W^{e,\lambda} \in D_d(X \times \mathbb{R})$ in the following way. For $x \in X$, the set $T_x = p^{-1}(x) \cap \operatorname{supp}\lambda$ is finite. Choose a connected neighborhood $U_x \subseteq X$ of x, and extend to a (unique) embedding $T_x \times U_x \to E$ over X, such that $p^{-1}(U_x) \cap \operatorname{supp}\lambda$ is contained in $T_x \times U_x$ (this can be done because $p_{|\operatorname{supp}(\lambda)}$ is a closed map).*

Define $W^{e,\lambda}_{|U_x} \in D_d(U_x \times \mathbb{R})$ as the construction in Definition 6.8 applied to the restriction of e to $T_x \times U_x$. (If $T_x = \emptyset$ then $W^{e,\lambda}_{|U_x} = W_{|U_x}$.) These elements agree on overlaps, so by the sheaf property of D_d we have defined $W^{e,\lambda} \in D_d(X \times \mathbb{R})$.

We are now ready to prove that $\beta D^{\pitchfork}_{d,\partial} \to D_d$ is a homotopy equivalence. It suffices to prove that any element of $D_d(X)$ is concordant to an element which lifts to $\beta D^{\pitchfork}_{d,\partial}(X)$ (plus corresponding relative statement).

Given an element $(W, \pi, f) \in D_d(X)$, we choose (as in the proof of Proposition 4.2) a locally finite open covering $X = \cup_j E_j$ and corresponding numbers $a_j \in \mathbb{R}$ such that $(W, a_j)_{|E_j} \in D^{\pitchfork}_d(E_j)$ for all j. We can assume that the a_j are all distinct constants.

For each pair j, k with $a_j < a_k$, let $E_{jk} = E_j \cap E_k$. Then $\varphi_{jk} = (W, a_j, a_k)_{|E_{jk}}$ is a morphism in $D^{\pitchfork}_d(E_{jk})$. We can assume that E_{jk} is either contractible or empty, so $(\pi, f)^{-1}(E_{jk} \times [a_j, a_k]) \cong E_{jk} \times W_0$ for a compact manifold W_0 with boundary. Consider the inclusion

$$(\pi, f)^{-1}(E_{jk} \times \{a_k\}) \to (\pi, f)^{-1}(E_{jk} \times [a_j, a_k]).$$

If this is π_0-surjective, then $\varphi_{jk} \in D^{\pitchfork}_{d,\partial}(E_{jk})$. If not, we can choose a finite set T_{jk} and an embedding $\tilde{e}_{jk} : E_{jk} \times T_{jk} \to (\pi, f)^{-1}(E_{jk} \times (a_j, a_k))$ over E_{jk} such that

$$(\pi, f)^{-1}(E_{jk} \times \{a_k\}) \amalg E_{jk} \times T_{jk} \to (\pi, f)^{-1}(E_{jk} \times [a_j, a_k])$$

is π_0-surjective. Let $r_{jk} = f \circ \tilde{e}_{jk} : E_{jk} \times T_{jk} \to \mathbb{R}$. Let $E = \coprod E_{jk} \times T_{jk}$, and let $(p, r) : E \to X \times \mathbb{R}$ be the resulting map. Then the $\tilde{e}_{jk}$ assemble to a map $\tilde{e} : E \to W$ over $X \times \mathbb{R}$. By possibly changing the f-level of $\tilde{e}_{jk}$, we can arrange that the various $\tilde{e}_{jk}$ have disjoint images so that $\tilde{e}$ is an embedding. E has contractible components, so the normal bundle of $\tilde{e}$ can be trivialized. Thus $\tilde{e}$ extends to an embedding $e : E \times \mathbb{R}^d \to W$ over X.

Now, for each $v \in p^{-1}(x) \subseteq E$, e defines an embedding $e_v : \{v\} \times \mathbb{R}^d \to W_x$, but $f_x : W_x \to \mathbb{R}$ might not be constant on the image of e_v. However, let $\varphi : [0, \infty) \to [0, \infty)$ be a smooth proper function with $\varphi[0, 1] = 0$ and $\varphi'(t) > 0$ for $t > 1$ and $\varphi(t) = t$ for $t \geq 2$. Then $f_x \circ e_v\big(\varphi(|u|)u\big)$ is constantly equal to $r(v)$

for $u \in D^d$ and agrees with $f_x e_v(u)$ outside $2D^d$. After changing f_x on the image of e_v and then re-choosing the embedding e (precompose it with an embedding of $\mathbb{R}^d$ into D^d), we can assume that e_v maps into $f_x^{-1}(r(v))$. This process works equally well in the parametrized setting, so after modifying $f : W \to \mathbb{R}$ we can assume that $e : E \times \mathbb{R}^d \to W$ is an embedding with $\pi \circ e(v,u) = p(v)$ and $f \circ e(v,u) = r(v)$. Choose compactly supported $\lambda_j : E_j \to [0,1]$ such that X is covered by the sets $\tilde{E}_j = \operatorname{Int} \lambda_j^{-1}(1)$, and let $\lambda_{jk} = \lambda_j \lambda_k : E_{jk} \to [0,1]$. These assemble to a function $\lambda : E \to \mathbb{R}$ with $p_{|\operatorname{supp}(\lambda)}$ proper.

Using these p, r, e and λ, Construction 6.10 provides an element $W^{e,\lambda} \in D_d(X \times \mathbb{R})$. We claim that $W_1^{e,\lambda} = W^{e,\lambda}_{|X \times \{1\}}$ lifts to an element of $\beta D^{\pitchfork}_{d,\partial}(X)$. Indeed, for $x \in \tilde{E}_j$, choose $b_{xj} \in \mathbb{R}$ in a tube around a_j such that $(x,1) \in \hat{V}(b_{xj}, b_{xj})$. Choose a neighborhood U_{xj} such that $U_{xj} \times \{1\} \subseteq \hat{V}(b_{xj}, b_{xj})$. Then $(W_1^{e,\lambda}, b_{xj}, b_{xj})_{|U_{xj}}$ is an object of $D_d^{\pitchfork}(U_{xj})$. As before, refining the U_{xj} to a locally finite covering defines an element of $\beta D_d^{\pitchfork}(X)$ which in turn, by Corollary 6.9, is an element of $\beta D^{\pitchfork}_{d,\partial}(X)$.

7. Harer type stability and $\mathcal{C}_2$

[Til97] introduced a version $\mathcal{S}_b$ of the category $\mathcal{C}^+_{2,\partial}$ to prove that $\mathbb{Z} \times B\Gamma_{\infty,n}$ is homology equivalent to an infinite loop space. This used two properties of $\mathcal{S}_b$. Firstly that $\mathcal{S}_b$ is symmetric monoidal, and secondly that $\Omega B\mathcal{S}_b$ is homology equivalent to $\mathbb{Z} \times B\Gamma_{\infty,n}$. In this section we will prove that $\Omega B\mathcal{C}^+_{2,\partial}$ is homology equivalent to $\mathbb{Z} \times B\Gamma_{\infty,n}$, using a version of the argument from [Til97].

The original stability theorem, proved by J. Harer in [Har85] is about the homology of the oriented mapping class group. In the language used in this paper, it can be stated as follows. Consider an oriented surface $W_{g,n}$ of genus g with n boundary circles. There are inclusions $W_{g,n} \to W_{g+1,n}$ and $W_{g,n} \to W_{g,n-1}$ by adding the torus $W_{1,2}$ or the disk $W_{0,1}$ to one of the boundary circles. Let $\operatorname{Diff}^+(W, \partial)$ denote the group of orientation-preserving diffeomorphisms of W that restrict to the identity near the boundary, and let

$$B\operatorname{Diff}^+(W_{g,n}; \partial) \to B\operatorname{Diff}^+(W_{g+1,n}; \partial), \tag{7.1}$$

$$B\operatorname{Diff}^+(W_{g,n}; \partial) \to B\operatorname{Diff}^+(W_{g,n-1}; \partial), \tag{7.2}$$

be the maps of classifying spaces induced from the above inclusions. Harer's stability theorem is that the maps in (7.1) and (7.2) induce isomorphisms, in integral homology in a range of dimensions that tends to infinity with g. (The range is approximately $g/2$ [Iva89].)

In the setup of chapter 5, Harer's stability theorem concerns the case $\theta : B \to BO(2)$, where $B = EO(2) \times_{O(2)} (O(2)/SO(2))$. Recently, homological stability theorems have been proved for surfaces with tangential structure in a number of other situations, which we now list.

- N. Wahl considered stability for non-orientable surfaces in [Wah06]. Let $S_{g,n}$ denote the connected sum of g copies of $\mathbb{R}P^2$ with n disks cut out, and consider the analogue of (7.1) with $\mathrm{Diff}^+(W_{g,n};\partial)$ replaced by $\mathrm{Diff}(S_{g,n};\partial)$. She proves a stability range (approximately $g/4$) for the associated mapping class groups $\pi_0\,\mathrm{Diff}(S_{g,n};\partial)$ and, using the contractibility of the component $\mathrm{Diff}_1(S_{g,n};\partial)$, deduces the homological stability for $B\,\mathrm{Diff}(S_{g,n};\partial)$.

- Stability for spin mapping class groups was established in [Har90] and [Bau04]. It corresponds to the category $\mathcal{C}_2^\theta$, with the tangential structure $\theta : B\,\mathrm{Spin}(2) \to BO(2)$, cf. [Gal06].
- Our final example is the stability theorem from [CM06], corresponding to the tangential structure

$$\theta : EO(2) \times_{O(2)} ((O(2)/SO(2)) \times Z) \to BO(2),$$

  where Z is a *simply connected* space.

With the above examples in mind, we now turn to a discussion of abstract stability in a topological category $\mathcal{C}$. We first remind the reader that a square diagram of spaces

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"] & X_0 \arrow[d, "g"] \\ X_1 \arrow[r, "p"] & X \end{tikzcd} \tag{7.3}$$

is *homotopy cartesian* if for all $x \in X_1$ the induced map of the vertical homotopy fibers

$$\underset{x}{\mathrm{hofib}}(f) \to \underset{p(x)}{\mathrm{hofib}}(g) \tag{7.4}$$

is a weak equivalence. Similarly, the diagram (7.3) is *homology cartesian* if (7.4) is a homology equivalence, i.e. induces an isomorphism in integral homology. If the map g is a Serre fibration, then diagram (7.3) is homotopy cartesian if it is cartesian.

We also remind the reader that if $\mathcal{C}$ is a category, then a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ determines, and is determined by, a category $(F \wr \mathcal{C})$ and a projection functor $(F \wr \mathcal{C}) \to \mathcal{C}$, such that the diagram of sets

$$\begin{tikzcd} N_1(F \wr \mathcal{C}) \arrow[r, "d_i"] \arrow[d] & N_0(F \wr \mathcal{C}) \arrow[d] \\ N_1\mathcal{C} \arrow[r, "d_i"] & N_0\mathcal{C} \end{tikzcd} \tag{7.5}$$

is cartesian for $i = 0$ (so d_i is the target map). Explicitly, $(F \wr \mathcal{C})$ is defined by

$$N_0(F \wr \mathcal{C}) = \{(x, c) \mid c \in N_0\mathcal{C}, x \in F(c)\},$$
$$N_1(F \wr \mathcal{C}) = \{(x, f) \mid f \in N_1\mathcal{C}, x \in F(d_0 c)\}.$$

Similarly, a functor F with values in the category of spaces determines, and is determined by, a *topological* category $(F \wr \mathcal{C})$ with a projection functor to $\mathcal{C}$ such that the diagram (7.5) is a cartesian diagram of spaces for $i = 0$. If the category

$\mathcal{C}$ itself is topological, then it is better to take this as a *definition*: A functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Spaces}$ is a topological category $(F \wr \mathcal{C})$ together with a functor $(F \wr \mathcal{C}) \to \mathcal{C}$ such that the diagram (7.5) is a cartesian diagram of spaces for $i = 0$.

We return to (7.5) under the assumption that the right hand vertical map is a Serre fibration. Then the diagram is homotopy cartesian for $i = 0$. It is homotopy cartesian also for $i = 1$, precisely if every morphism $f : x \to y$ in $\mathcal{C}$ induces a weak equivalence $F(f) : F(x) \to F(y)$. Similarly it is homology cartesian for $i = 1$, precisely if every $f : x \to y$ induces an isomorphism $F(f)_* : H_*(F(x)) \to H_*(F_y)$.

Proposition 7.1. *Let $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Spaces}$ be a functor such that $N_0(F \wr \mathcal{C}) \to N_0\mathcal{C}$ is a Serre fibration. Suppose that every $f : x \to y$ in $\mathcal{C}$ induces an isomorphism $F(f)_* : H_*(F(x)) \to H_*(F(y))$ and that $B(F \wr \mathcal{C})$ is contractible. Then for each object $c \in \mathcal{C}$ there is a map*

$$F(c) \to \Omega_c B\mathcal{C}$$

which induces an isomorphism in integral homology.

Proof. The assumptions imply that diagram (7.5) is homology cartesian for $i = 0$ and $i = 1$, and by induction every diagram of the form

$$\begin{tikzcd} N_k(F \wr \mathcal{C}) \arrow[r, "d_i"] \arrow[d] & N_{k-1}(F \wr \mathcal{C}) \arrow[d] \\ N_k\mathcal{C} \arrow[r, "d_i"] & N_{k-1}\mathcal{C} \end{tikzcd}$$

is homology cartesian. Then it follows from [MS76, Proposition 4] that the diagram

$$\begin{tikzcd} N_0(F \wr \mathcal{C}) \arrow[r, "d_i"] \arrow[d] & B(F \wr \mathcal{C}) \arrow[d] \\ N_0\mathcal{C} \arrow[r, "d_i"] & B\mathcal{C} \end{tikzcd}$$

is homology cartesian, i.e. the induced map of vertical homotopy fibers is a homology isomorphism. Let $c \in \mathrm{Ob}\mathcal{C}$. Since $N_0(F \wr \mathcal{C}) \to N_0\mathcal{C}$ is assumed a Serre fibration, the homotopy fiber at c of the left vertical map is $F(c)$. Since $B(F \wr \mathcal{C})$ is assumed contractible, the homotopy fiber of the right vertical map at c is $\Omega_c B\mathcal{C}$. □

We apply this in the case where $\mathcal{C} \subseteq \mathcal{C}_{\theta,\partial}$ is the subcategory of objects (M, a) with $a < 0$, and $\theta : B \to BO(2)$ is a tangential structure for which we have a Harer type stability theorem. To define a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Spaces}$, let $S^1 \subseteq \mathbb{R}^{2-1+\infty}$ be a fixed circle, and consider the objects $b_i = \{i\} \times S^1$ in $(\mathcal{C}_{\theta,\partial})$, $i \in \mathbb{N}$. Choose morphisms $\beta_i \subseteq [i, i+1] \times \mathbb{R}^{2-1+\infty}$ from b_i to b_{i+1} which are connected surfaces of genus 1, and compatible θ-structures on the b_i and the β_i. We use here that the tangent bundle of the surface $\beta_i \cong W_{1,2}$ can be trivialized. Let $F_i : \mathcal{C}\mathrm{op} \to \mathrm{Spaces}$ be the functors

$$F_i(c) = \mathcal{C}_{\theta,\partial}(c, b_i)$$

and let

$$F(c) = \operatorname{hocolim}(F_0(c) \xrightarrow{\circ\beta_0} F_1(c) \xrightarrow{\circ\beta_1} \cdots).$$

As a space, $N_0(F_i \wr \mathcal{C})$ is defined by the cartesian diagram

$$\begin{tikzcd} N_0(F_i \wr \mathcal{C}) \arrow[r] \arrow[d] & X_1 \arrow[d, "(d_0,d_1)"] \\ N_0\mathcal{C} \arrow[r, "(b_i,\mathrm{id})"] & X_0 \times N_0\mathcal{C}, \end{tikzcd}$$

where $X_1 = \{(W, a_0, a_1, l) \in N_1\mathcal{C}_{\theta,\partial} \mid a_0 < 0 < a_1\}$ and $X_0 = \{(M, a, l) \in N_0\mathcal{C}_{\theta,\partial} \mid a > 0\}$. It follows from [KM97] that the right hand vertical map is a smooth Serre fibration, so $N_0(F_i \wr \mathcal{C}) \to N_0\mathcal{C}$ and in turn $N_0(F \wr \mathcal{C}) \to N_0\mathcal{C}$ are Serre fibrations, as required in Proposition 7.1. The category $(F_i \wr \mathcal{C})$ has terminal object id_{b_i}, so $B(F_i \wr \mathcal{C})$ is contractible. Therefore $B(F \wr \mathcal{C}) = \operatorname{hocolim}_i B(F_i \wr \mathcal{C})$ is also contractible. Finally, if $c = \{t\} \times S_n$, where $S_n \subseteq \mathbb{R}^{2-1+\infty}$ is a disjoint union of n circles, then the homotopy equivalence (5.4) gives

$$F_i(c) \simeq \coprod_{g \geq 0} E\,\mathrm{Diff}(W_{g,n+1}, \partial) \times_{\mathrm{Diff}(W_{g,n+1},\partial)} \mathrm{Bun}^{\partial}(TW_{g,n+1}, \theta^* U_d),$$

where $W_{g,n+1}$ is a surface of genus g with $n+1$ boundary components, and $\mathrm{Diff}(W_{g,n+1}, \partial)$ is the topological group of diffeomorphisms of $W_{g,n+1}$ restricting to the identity near the boundary.

Any morphism $x \to y$ in $\mathcal{C}$ induces a map $F_i(x) \to F_i(y)$ which corresponds to including one connected surface W into another connected surface. After taking the limit $g \to \infty$, any morphism $x \to y$ in $\mathcal{C}$ induces an isomorphism $H_*(F(x)) \to H_*(F(y))$ in the four case listed above, cf. [Gal06], [CM06], [Wah06]. In the case of ordinary orientations we get

$$F(c) \simeq \mathbb{Z} \times B\Gamma_{\infty,n+1},$$

so we get a new proof of the generalized Mumford conjecture.

Theorem 7.2 ([MW02])**.** *There is a homology equivalence*

$$\alpha : \mathbb{Z} \times B\Gamma_{\infty,n} \to \Omega^\infty MT(2)^+.$$

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