A Short Exposition of the Madsen-Weiss Theorem

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The theorem of Madsen and Weiss [MW] identifies the homology of mapping class groups of surfaces, in a stable dimension range, with the homology of a certain infinite loopspace. This result is not only intrinsically interesting, showing that two objects that appear to be quite different turn out to be homologically equivalent, but it also allows explicit calculations of the stable homology, rather easily for rational coefficients and with some work for mod p coefficients. Outside the stable dimension range the homology of mapping class groups appears to be quite complicated and is still very poorly understood, so it is surprising that there is such a simple and appealing description in the stable range.

The Madsen-Weiss theorem has a very classical flavor, and in retrospect it seems that it could have been proved in the 1970s or 1980s since the main ingredients were available then. However, at that time it was regarded as very unlikely that the stable homology of mapping class groups could be that of an infinite loopspace. The initial breakthrough came in a 1997 paper of Tillmann [T] where this unexpected result was proved. It remained then to determine whether the infinite loopspace was a familiar one. A conjecture in this direction was made in a 2001 paper of Madsen and Tillmann [MT], with some supporting evidence, and this conjecture became the Madsen-Weiss theorem.

The original proof of the Madsen-Weiss theorem was rather complicated, but major simplifications have been found since then. The purpose of the present paper is to explain a proof that uses a number of these later simplifications, particularly some due to Galatius and Randal-Williams which make the proof really quite elementary, apart from the three main classical ingredients:

(1) The fact that the diffeomorphism group of a closed orientable surface has contractible components once the genus of the surface is at least 2. This is originally a theorem of Earle-Eells [EE] proved by analytic methods, but a purely topological proof was found soon after by Gramain [Gr]. Also needed is the extension of this result to compact orientable surfaces with boundary, originally a result in [ES], but proved in the Gramain paper as well (and easily deducible by topological methods from the Earle-Eells result).

(2) Harer stability [Har], the fact that the ith homology group of the mapping class group of a compact orientable surface is independent of the genus and the number of boundary components, once the genus is sufficiently large with respect to i. Subsequent improvements in the stable dimension range were made by Ivanov [I], Boldsen [B], and Randal-Williams [RW], and a significant gap in Harer's proof was filled by Wahl [W1]. For a full exposition of the current state of the art on this theorem see [W2].

(3) The Group Completion Theorem from around 1970. A nice exposition of this fundamental result in algebraic topology was published in [MS], and an illuminating general discussion can be found in [A]. The original version seems to be due to Barratt and Priddy [BP].

The strategy we follow, as in [GRW], is to use (3) to prove a theorem that is independent of (1) and (2). This theorem can be phrased as saying that the classifying space of the group of compactly supported diffeomorphisms of an infinite-genus surface is homologically equivalent to a certain infinite loopspace (or more properly, one component of this infinite loopspace). Then one can quote (1) and (2) to restate this result in terms of the homology of mapping class groups of compact surfaces in the stable range.

The Scanning Map

Let us describe the main idea underlying the Madsen-Weiss theorem, which leads to the infinite loopspace mentioned above. The roots of this idea can be traced back a long way, to the Pontryagin-Thom construction around 1950. Another version of the idea appeared later in the 1970s in papers by Segal [S1, S2] and McDuff [M]. A more immediate predecessor is the paper of Madsen-Tillmann [MT] which explicitly conjectures the Madsen-Weiss theorem.

The homology of the mapping class group of a closed oriented surface S is the homology of an Eilenberg-MacLane space $K(\Gamma,1)$, with Γ the mapping class group. We are free to choose any $K(\Gamma,1)$ we like, and there is a particular choice that works well for the Madsen-Weiss theorem. This is the space $\mathcal{C}(S,\mathbb{R}^\infty)$ of all smooth oriented subsurfaces of $\mathbb{R}^\infty$ diffeomorphic to S. The symbol $\mathcal{C}$ is chosen to indicate that $\mathcal{C}(S,\mathbb{R}^\infty)$ is the space of all possible "configurations" of S in $\mathbb{R}^\infty$. As we explain in an Appendix, there is a fiber bundle $\mathcal{E}(S,\mathbb{R}^\infty)\to\mathcal{C}(S,\mathbb{R}^\infty)$ where $\mathcal{E}(S,\mathbb{R}^\infty)$ is the space of smooth embeddings $S\to\mathbb{R}^\infty$. The fiber of this bundle is the group $\mathrm{Diff}^+(S)$ of orientation-preserving diffeomorphisms of S. The total space $\mathcal{E}(S,\mathbb{R}^\infty)$ is contractible, so the long exact sequence of homotopy groups for the bundle gives isomorphisms $\pi_i\mathrm{Diff}^+(S)\cong\pi_{i+1}\mathcal{C}(S,\mathbb{R}^\infty)$ for all i. By the Earle-Eells theorem these groups are trivial for $i>0$, at least when the genus of S is at least 2, so we see that $\mathcal{C}(S,\mathbb{R}^\infty)$ is a $K(\Gamma,1)$ for the mapping class group in these cases.

To place things in their natural setting, consider a smooth closed orientable manifold M of arbitrary dimension, and let $\mathcal{C}(M,\mathbb{R}^\infty)$ be the space of all smooth oriented submanifolds of $\mathbb{R}^\infty$ diffeomorphic to M. This is the union of its subspaces $\mathcal{C}(M,\mathbb{R}^n)$ of smooth oriented submanifolds of $\mathbb{R}^n$ diffeomorphic to M, for finite values of n, with the direct limit topology, where each $\mathcal{C}(M,\mathbb{R}^n)$ is given the usual C^∞ topology as the orbit space of the embedding space $\mathcal{E}(M,\mathbb{R}^n)$ under the action of $\mathrm{Diff}^+(M)$ by composition. By definition, in the direct limit topology on $\mathcal{C}(M,\mathbb{R}^\infty)$ a set is open if and only if it intersects each subspace $\mathcal{C}(M,\mathbb{R}^n)$ in an open set. A key feature of direct limit topologies is that compact subspaces lie in finite stages of the direct limit, so the homotopy groups and homology groups of a direct limit are the direct limits of the homotopy and homology groups of the finite stages.

It is natural to ask what information can be extracted from an embedded submanifold $M \subset \mathbb{R}^n$ of dimension m just by looking locally. Imagine taking a powerful magnifying lens and moving it all around $\mathbb{R}^n$ to see what the submanifold M looks like. For most positions of the lens one is not close enough to M to see anything of M at all, so one sees just the empty set, but as one moves near M one sees a small piece of M that appears to be almost flat. Moving toward M, this piece first appears at the edge of the lens, then moves to the center. The space of almost flat m-planes in an n-ball has the same homotopy type as the subspace of actually flat m-planes since one can canonically deform almost flat planes to their tangent planes at their centers of mass. Regarding the n-ball as $\mathbb{R}^n$, we thus have, for each position of the lens where the view of M is nonempty, an m-plane in $\mathbb{R}^n$. This plane has an orientation determined by the given orientation of M. We will use the notation $AG_{n,m}$ for the *affine Grassmannian* of oriented flat m-planes in $\mathbb{R}^n$, where the word "affine" indicates that the planes need not pass through the origin. Taking into account positions of the lens where the view of M is empty, we then have a point in the one-point compactification $AG^+_{n,m}$ of $AG_{n,m}$ for each position of the lens. Positions of the lens near infinity in $\mathbb{R}^n$ give an empty view of M, hence map to the compactification point at infinity in $AG^+_{n,m}$. Thus by letting the position of the lens vary throughout all of $\mathbb{R}^n$ we obtain a point in the n-fold loopspace $\Omega^n AG^+_{n,m}$, where the basepoint of $AG^+_{n,m}$ is taken to be the point at infinity. This point in $\Omega^n AG^+_{n,m}$ associated to the submanifold $M \subset \mathbb{R}^n$ depends on choosing a sufficiently large power of magnification for the lens, which in turn can depend on the embedding of M in $\mathbb{R}^n$. Making the plausible assumption that the magnification can be chosen to vary continuously with M, we then obtain a map $\mathcal{C}(M,\mathbb{R}^n) \to \Omega^n AG^+_{n,m}$. Letting n increase, the natural inclusion $\mathcal{C}(M,\mathbb{R}^n) \hookrightarrow \mathcal{C}(M,\mathbb{R}^{n+1})$ corresponds to the inclusion $\Omega^n AG^+_{n,m} \hookrightarrow \Omega^{n+1} AG^+_{n+1,m}$ obtained by applying Ω^n to the inclusion $AG^+_{n,m} \hookrightarrow$

$\Omega AG^+_{n+1,m}$ that translates an m-plane in $\mathbb{R}^n$ from $-\infty$ to $+\infty$ in the $(n+1)$st coordinate of $\mathbb{R}^{n+1}$. Passing to the limit over n, we get a map

$$\mathcal{C}(M,\mathbb{R}^\infty) \to \Omega^\infty AG^+_{\infty,m}$$

which we will refer to as the *scanning map*. (In the case $m=0$ when M is a finite set of points, the scanning process is described explicitly in [S2], which seems to be the first place where the term "scanning" is used in this context.)

There is another way to describe the scanning map just in terms of tangent planes. For this it is convenient to replace $\mathcal{C}(M,\mathbb{R}^n)$ by the homotopy equivalent space $\mathcal{C}'(M,\mathbb{R}^n)$ of oriented smooth submanifolds of $\mathbb{R}^n$ diffeomorphic to M together with a choice of tubular neighborhood and an identification of this neighborhood with the normal bundle of M in $\mathbb{R}^n$. For the tubular neighborhoods we can restrict attention just to ϵ-neighborhoods for sufficiently small ϵ, where the fibers of the normal bundle are open ϵ-disks normal to M, reparametrized as vector spaces by some canonical rescaling of the lengths of normal vectors. Denote such a normal bundle neighborhood as $p\colon E \to M$. The scanning map $\mathcal{C}'(M,\mathbb{R}^n) \to \Omega^n AG^+_{n,m}$ in this context sends the complement of E to the basepoint, the point at infinity in $AG^+_{n,m}$, and sends a point x in E to the tangent plane to M at $p(x)$, first translated to the origin in $\mathbb{R}^n$ then translated by the vector from x to $p(x)$ in the fiber of E over $p(x)$. Since we have rescaled vectors in this fiber, the length of the vector from x to $p(x)$ approaches infinity as x approaches the complement of E. Thus we have a continuous map $\mathcal{C}'(M,\mathbb{R}^n) \to \Omega^n AG^+_{n,m}$. Letting n go to infinity as before, we get a scanning map $\mathcal{C}'(M,\mathbb{R}^\infty) \to \Omega^\infty AG^+_{\infty,m}$.

The affine Grassmannian $AG_{n,m}$ can be described in terms of the usual Grassmannian $G_{n,m}$ of oriented m-planes through the origin in $\mathbb{R}^n$. The projection $AG_{n,m} \to G_{n,m}$ translating each m-plane to the parallel plane through the origin is a vector bundle whose fiber over a given m-plane P is the vector space of vectors orthogonal to P since there is a unique such vector translating P to any given plane parallel to P. The vector bundle $AG_{n,m}$ is thus the orthogonal complement of the canonical bundle over $G_{n,m}$, and the one-point compactification $AG^+_{n,m}$ is the Thom space of this complementary vector bundle.

For the scanning map $\mathcal{C}(M,\mathbb{R}^\infty) \to \Omega^\infty AG^+_{\infty,m}$ the source space depends on the manifold M but the target space does not, so one would hardly expect this map to be any sort of equivalence for arbitrary M. One might have a better chance if one could replace the source by some sort of amalgam or limit over all choices of M, and this is what the Madsen-Weiss theorem does in the case of surfaces, when $m=2$, using the fact that there are not too many different choices for connected orientable closed surfaces M, just one for each genus. The simplest infinite-genus surface S_∞ can be regarded as the union of an increasing sequence of surfaces $S_{g,1}$ of genus g with one boundary component. We can then consider the group $\mathrm{Diff}_c(S_\infty)$ of compactly supported diffeomorphisms of S_∞. Note that these are automatically orientation-preserving. One can regard $\mathrm{Diff}_c(S_\infty)$ as the union or direct limit of the groups $\mathrm{Diff}(S_{g,1})$ of diffeomorphisms of $S_{g,1}$ that restrict to the identity on the boundary circle (and whose derivatives of all orders equal those of the identity diffeomorphism at points in the boundary) under the natural inclusions $\mathrm{Diff}(S_{g,1}) \hookrightarrow \mathrm{Diff}(S_{g+1,1})$ induced by the standard embedding $S_{g,1} \hookrightarrow S_{g+1,1}$, extending diffeomorphisms by the identity on the complement of $S_{g,1}$ in $S_{g+1,1}$. Part of Harer stability is the fact that mapping class groups of closed surfaces and surfaces with boundary have isomorphic homology in the stable dimension range, so there is no harm in considering only the surfaces $S_{g,1}$ instead of closed surfaces.

Let $\mathcal{C}_g(S_\infty, \mathbb{R}^\infty)$ be the space of subsurfaces of $\mathbb{R}^\infty$ diffeomorphic to S_∞ that agree with a fixed properly embedded copy of S_∞ outside $S_{g,1}$. This space is a $K(\Gamma,1)$ for $\Gamma = \pi_0\mathrm{Diff}(S_{g,1})$. There are inclusions $\mathcal{C}_g(S_\infty,\mathbb{R}^\infty) \subset \mathcal{C}_{g+1}(S_\infty,\mathbb{R}^\infty)$, and we let $\mathcal{C}(S_\infty,\mathbb{R}^\infty) = \cup_g \mathcal{C}_g(S_\infty,\mathbb{R}^\infty)$ with the direct limit topology. The basic properties of the direct limit topology imply that $\mathcal{C}(S_\infty,\mathbb{R}^\infty)$ is a $K(\Gamma,1)$ for $\Gamma = \pi_0\mathrm{Diff}_c(S_\infty)$, the direct limit of the mapping class groups $\pi_0\mathrm{Diff}(S_{g,1})$.

The version of the Madsen-Weiss theorem that we will prove is the following:

The Madsen-Weiss Theorem. *There is an isomorphism*

$$H_*(\mathcal{C}(S_\infty,\mathbb{R}^\infty)) \cong H_*(\Omega^\infty_0 AG^+_{\infty,2})$$

where $\Omega^\infty_0 AG^+_{\infty,2}$ denotes the basepoint path-component of $\Omega^\infty AG^+_{\infty,2}$.

The reason for replacing Ω^∞ by Ω^∞_0 is that $\mathcal{C}(S_\infty,\mathbb{R}^\infty)$ is path-connected since each $\mathcal{C}_g(S_\infty,\mathbb{R}^\infty)$ is path-connected. All the path-components of $\Omega^\infty AG^+_{\infty,2}$ have the same homotopy type since it is an H-space with π_0 a group, so it does not matter which path-component we choose.

One might wish to have a scanning map $\mathcal{C}(S_\infty,\mathbb{R}^\infty) \to \Omega^\infty AG^+_{\infty,2}$ that induces the homology isomorphism in the theorem, but constructing a scanning map like this for noncompact manifolds would require an additional argument and this is not actually needed for the proof of the theorem. Instead the isomorphism in the theorem will be obtained in a somewhat less direct fashion by recasting the scanning idea in a slightly different form.

A Convention: We will sometimes say a map is a homotopy equivalence but only prove it is a weak homotopy equivalence, inducing isomorphisms on all homotopy groups. This will not be a problem since in the end we are only interested in homology groups and weak homotopy equivalences induce isomorphisms on homology.

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1. A Warm-up Case: The Symmetric Group

To give the idea of how the scanning method works in a simple case let us show that the infinite symmetric group $\Sigma_\infty = \cup_p \Sigma_p$ has the same homology as one component of $\Omega^\infty S^\infty$, originally a theorem of Barratt, Priddy, and Quillen.

There is an easy classical construction of a classifying space for Σ_p as the space $\mathcal{C}_p(\mathbb{R}^\infty)$ of all configurations of p distinct unordered points in $\mathbb{R}^\infty$. This is a quotient space of the subspace $\widetilde{\mathcal{C}}_p(\mathbb{R}^\infty) \subset (\mathbb{R}^\infty)^p$ consisting of all ordered p-tuples of distinct points in $\mathbb{R}^\infty$. The group Σ_p acts freely on $\widetilde{\mathcal{C}}_p(\mathbb{R}^\infty)$ by permuting the points in an ordered p-tuple, so one has a covering space $\widetilde{\mathcal{C}}_p(\mathbb{R}^\infty) \to \mathcal{C}_p(\mathbb{R}^\infty)$. It follows that $\mathcal{C}_p(\mathbb{R}^\infty)$ is a classifying space $B\Sigma_p$ since $\widetilde{\mathcal{C}}_p(\mathbb{R}^\infty)$ is contractible by an elementary argument. Namely, the identity map of $\mathbb{R}^\infty$ is linearly isotopic to the embedding $(x_1, x_2, \cdots) \to (0, x_1, x_2, \cdots)$, and this isotopy induces a deformation of ordered configurations in $\mathbb{R}^\infty$ to ordered configurations in the subspace $\{0\} \times \mathbb{R}^\infty$, where such configurations are linearly isotopic to a fixed standard configuration in $\mathbb{R} \times \{0\}$.

The space $\mathcal{C}_p(\mathbb{R}^\infty)$ of p-element subsets of $\mathbb{R}^\infty$ is the union of its subspaces $\mathcal{C}_p(\mathbb{R}^n)$ of p-element subsets of $\mathbb{R}^n$. Taking the union over all $p \geq 0$ we have the space $\mathcal{C}(\mathbb{R}^n) = \coprod_p \mathcal{C}_p(\mathbb{R}^n)$. The subspaces $\mathcal{C}_p(\mathbb{R}^n)$ are the path-components of $\mathcal{C}(\mathbb{R}^n)$.

There is another topology on $\mathcal{C}(\mathbb{R}^n)$ that will be of interest. In this topology convergence will mean convergence in larger and larger compact sets rather than convergence everywhere. A basis for this topology consists of sets $C(B, V)$ where:

(i) B is a closed ball in $\mathbb{R}^n$ centered at the origin.

(ii) V is an open set of configurations of p points in the interior of B for some $p \geq 0$, using the usual topology on configurations of p points in a ball.

(iii) $C(B, V)$ consists of all configurations c disjoint from ∂B such that $c \cap B$ is in V.

Letting B and V vary, it is easy to check that the sets $C(B, V)$ give a basis for a topology on the set $\mathcal{C}(\mathbb{R}^n)$. The notation we use for $\mathcal{C}(\mathbb{R}^n)$ with this new topology is $\mathcal{C}^n$. It would not materially affect any of our subsequent arguments if $\mathcal{C}^n$ were enlarged to include all closed discrete subsets of $\mathbb{R}^n$, finite or infinite. However, there seems to be no advantage in doing this, so we will keep the original definition of $\mathcal{C}^n$.

Unlike $\mathcal{C}(\mathbb{R}^n)$, the space $\mathcal{C}^n$ is path-connected when $n > 0$ since any configuration C can be connected to the empty configuration by the path expanding C radially from any point in the complement of C, with the expansion factor increasing from 1 to ∞. This path is continuous since in any ball B it is eventually constant, the empty configuration.

In fact the homotopy type of $\mathcal{C}^n$ is easy to determine completely:

Proposition 1.1. *$\mathcal{C}^n$ is homotopy equivalent to the sphere S^n, viewed as the subspace of $\mathcal{C}^n$ consisting of configurations with at most one point.*

Proof. We show that $\pi_i(\mathcal{C}^n, S^n) = 0$ for all i. Represent an element of this relative homotopy group by a family of configurations C_t, $t \in D^i$. Since D^i is compact, there is a minimum distance between the points of C_t in a bounded subset of $\mathbb{R}^n$ as t ranges over D^i, so we can choose a small ball B about the origin that contains at most one point of C_t for all t. Expanding B radially to $\mathbb{R}^n$ and simultaneously expanding C_t gives a deformation of C_t to a family in S^n, staying in S^n for $t \in \partial D^i$. □

Every configuration C in $\mathcal{C}(\mathbb{R}^n)$ can be connected to the empty configuration by the path in $\mathcal{C}^n$ obtained by translating C to infinity in a given direction, say in the direction of a unit vector v in $\mathbb{R}^n$. Letting v vary over all unit vectors, we are translating C to infinity in all directions, thereby obtaining a continuous family of configurations in $\mathcal{C}^n$ parametrized by $S^n = \mathbb{R}^n \cup \{\infty\}$. Thus C determines a map $S^n \to \mathcal{C}^n$ taking the basepoint ∞ of S^n to the empty configuration, which we take as the basepoint of $\mathcal{C}^n$. Such a map $S^n \to \mathcal{C}^n$ is a point in the n-fold loopspace $\Omega^n\mathcal{C}^n$. Letting the initial configuration C vary, we obtain a map $\mathcal{C}(\mathbb{R}^n) \to \Omega^n\mathcal{C}^n$.

As we saw in the proof of Proposition 1.1, configurations in $\mathcal{C}^n$ can be rescaled radially to their germs in a small disk around the origin. This leads to a deformation of the map $\mathcal{C}(\mathbb{R}^n) \to \Omega^n\mathcal{C}^n$ to a map

$$\mathcal{C}(\mathbb{R}^n) \to \Omega^n S^n \subset \Omega^n\mathcal{C}^n$$

which can be described as taking a configuration in $\mathcal{C}(\mathbb{R}^n)$ and scanning it by moving a small magnifying lens over every point in $\mathbb{R}^n$.

This scanning map $\mathcal{C}(\mathbb{R}^n) \to \Omega^n S^n$ cannot be a homotopy equivalence or even a homology equivalence, as one can see just by looking at what happens on π_0. The components of $\mathcal{C}(\mathbb{R}^n)$ are the subspaces $\mathcal{C}_p(\mathbb{R}^n)$ so $\pi_0\mathcal{C}(\mathbb{R}^n)$ is $\mathbb{Z}_{\geq 0}$, but $\pi_0\Omega^n S^n = \pi_n S^n$ is $\mathbb{Z}$, and the map $\pi_0\mathcal{C}(\mathbb{R}^n) \to \pi_0\Omega^n S^n$ is the inclusion $\mathbb{Z}_{\geq 0} \hookrightarrow \mathbb{Z}$. Furthermore the components of $\mathcal{C}(\mathbb{R}^n)$ cannot be homotopy equivalent to the corresponding components of $\Omega^n S^n$ since the components of $\Omega^n S^n$ are all homotopy equivalent to each other since $\Omega^n S^n$ is an H-space with π_0 a group, but the components of $\mathcal{C}(\mathbb{R}^n)$ have different homotopy types since $\pi_1 \mathcal{C}_p(\mathbb{R}^n)$ is Σ_p for $n \geq 3$, or the braid group B_p for $n = 2$. There are also differences in homology since $H_i(\Omega^n S^n)$ is known to be nonzero for infinitely many i when $n \geq 2$, so the same is true for each component of $\Omega^n S^n$, whereas $H_i \mathcal{C}_p(\mathbb{R}^n)$ is zero for sufficiently large i since $\mathcal{C}_p(\mathbb{R}^n)$ is a finite-dimensional manifold.

The idea of stabilizing $\mathcal{C}_p(\mathbb{R}^n)$ by letting p go to ∞ now enters the picture. There are inclusions $\mathcal{C}_p(\mathbb{R}^n) \hookrightarrow \mathcal{C}_{p+1}(\mathbb{R}^n)$ that can be defined in various ways, for example by taking a configuration in $\mathcal{C}_p(\mathbb{R}^n)$ and compressing its first coordinates to lie in $(-\infty, 0)$ rather than all of $\mathbb{R}$, then adjoining a $(p+1)$st point with positive first coordinate such as the point $(1, 0, \cdots, 0)$. Iterating these inclusions $\mathcal{C}_p(\mathbb{R}^n) \hookrightarrow \mathcal{C}_{p+1}(\mathbb{R}^n)$, we can then form the limit $\lim_p \mathcal{C}_p(\mathbb{R}^n)$. There is a corresponding stabilization in $\Omega^n S^n$, where the component $\Omega^n_p S^n$ consisting of degree p maps $S^n \to S^n$ injects into the component $\Omega^n_{p+1} S^n$ by summing with a degree 1 map. This stabilization $\Omega^n_p S^n \hookrightarrow \Omega^n_{p+1} S^n$ is a homotopy equivalence, with homotopy inverse given by summing with a degree -1 map, so when we form $\lim_p \Omega^n_p S^n$ we are in essence just identifying the various components $\Omega^n_p S^n$. The stabilizations in $\mathcal{C}(\mathbb{R}^n)$ and $\Omega^n S^n$ are compatible, so we get a map

$$\lim_p \mathcal{C}_p(\mathbb{R}^n) \to \lim_p \Omega^n_p S^n \simeq \Omega^n_0 S^n$$

We will show that this map induces an isomorphism on homology. Letting n go to infinity then yields an isomorphism $H_*(\lim_p \mathcal{C}_p(\mathbb{R}^\infty)) \cong H_*(\Omega^\infty_0 S^\infty)$, which is the Barratt-Priddy-Quillen theorem since $\mathcal{C}_p(\mathbb{R}^\infty)$ is a classifying space $B\Sigma_p$.

Define a filtration of $\mathcal{C}^n$ by subspaces $\mathcal{C}^{n,0} \subset \mathcal{C}^{n,1} \subset \cdots \subset \mathcal{C}^{n,n} = \mathcal{C}^n$ where $\mathcal{C}^{n,k}$ consists of the configurations contained in $\mathbb{R}^k \times (0,1)^{n-k}$. Thus elements of $\mathcal{C}^{n,k}$ can go to infinity in only k directions. The first space $\mathcal{C}^{n,0}$ is homeomorphic to $\mathcal{C}(\mathbb{R}^n)$ since $(0,1)^n$ is homeomorphic to $\mathbb{R}^n$.

There is a natural map $\mathcal{C}^{n,k} \to \Omega \mathcal{C}^{n,k+1}$ obtained by translating configurations from $-\infty$ to $+\infty$ in the $(k+1)$st coordinate.

Proposition 1.2. *The map $\mathcal{C}^{n,k} \to \Omega \mathcal{C}^{n,k+1}$ is a homotopy equivalence when $k > 0$. When $k = 0$ the map $\mathcal{C}^{n,0} \to \Omega \mathcal{C}^{n,1}$ induces a homology isomorphism $H_*(\lim_p \mathcal{C}^{n,0}_p) \to H_*(\Omega_0 \mathcal{C}^{n,1})$.*

Combining these maps and their iterated loopings gives a composition

$$\mathcal{C}^{n,0} \to \Omega \mathcal{C}^{n,1} \to \Omega^2 \mathcal{C}^{n,2} \to \cdots \to \Omega^n \mathcal{C}^{n,n}$$

which is the same as the earlier map $\mathcal{C}(\mathbb{R}^n) \to \Omega^n \mathcal{C}^n$ restricted to the subspace $\mathcal{C}^{n,0}$ of $\mathcal{C}(\mathbb{R}^n)$. By the proposition, this composition induces an isomorphism $H_*(\lim_p \mathcal{C}_p^{n,0}) \cong H_*(\Omega_0^n \mathcal{C}^{n,n})$, so the proof of the theorem will be complete once this proposition is proved.

The space $\mathcal{C}^{n,k}$ is an H-space when $k < n$, with the product given by juxtaposition in the $(k+1)$st coordinate, after compressing the interval $(0,1)$ in this coordinate to $(0,1/2)$ as in the definition of composition of loops. The homotopy-identity element for the H-space structure is the empty configuration, and the multiplication is homotopy-associative. Just as the Moore loopspace provides a monoid version of the usual loopspace, with strict associativity and a strict identity, there is a monoid version $\mathcal{M}^{n,k}$ of $\mathcal{C}^{n,k}$. This is the subspace of $\mathcal{C}^n \times [0,\infty)$ consisting of pairs (C,a) with C a configuration in $\mathbb{R}^k \times (0,a) \times (0,1)^{n-k-1}$. The product in $\mathcal{M}^{n,k}$ is again given by juxtaposition in the $(k+1)$st coordinate, but this time without any compression. The inclusion $\mathcal{C}^{n,k} \hookrightarrow \mathcal{M}^{n,k}$ as pairs $(C,1)$ is a homotopy equivalence.

A topological monoid $\mathcal{M}$ has a classifying space $B\mathcal{M}$. Let us recall the construction. In the special case that $\mathcal{M}$ has the discrete topology, $B\mathcal{M}$ is the Δ-complex having a single vertex, an edge for each element of $\mathcal{M}$, and more generally a p-simplex for each p-tuple $(m_1,\cdots,m_p)$ of elements of $\mathcal{M}$. The faces of such a simplex are the $(p-1)$-tuples obtained by deleting the first or last m_i or by replacing two adjacent m_i's by their product in $\mathcal{M}$. Thus $B\mathcal{M}$ is a quotient space of $\coprod_p \Delta^p \times \mathcal{M}^p$ with certain identifications over $\partial\Delta^p \times \mathcal{M}^p$ for each p. Essentially the same construction can be made when $\mathcal{M}$ has a nontrivial topology. One gives $\coprod_p \Delta^p \times \mathcal{M}^p$ the product topology and then forms a quotient using the same rules for identifications over $\partial\Delta^p \times \mathcal{M}^p$ for each p.

The main step in proving the previous proposition is the following:

Proposition 1.3. $\mathcal{C}^{n,k+1} \simeq B\mathcal{M}^{n,k}$.

Proof. We will define a map $\sigma : B\mathcal{M}^{n,k} \to \mathcal{C}^{n,k+1}$ and show this induces isomorphisms on all homotopy groups. A point in $B\mathcal{M}^{n,k}$ is given by a p-tuple of points $m_1,\cdots,m_p$ in $\mathcal{M}^{n,k}$ and a point in Δ^p with barycentric cooordinates $w_0,\cdots,w_p$. When we form the product $m_1\cdots m_p$ we obtain configurations $C_1,\cdots,C_p$ whose $(k+1)$st coordinates lie in intervals $[a_0,a_1],[a_1,a_2],\cdots,[a_{p-1},a_p]$ for $0 = a_0 \le a_1 \le \cdots \le a_p$. The union of the configurations C_i is a configuration $C \in \mathcal{C}^{n,k+1}$. However, C does not depend continuously on the given point of $B\mathcal{M}^{n,k}$ since as the weight w_0 or w_p goes to 0, the configuration C_1 or C_p is suddenly deleted from C. Furthermore, when w_0 goes to 0 the remaining configuration is suddenly translated a distance a_1 to the left in the $(k+1)$st coordinate of $\mathbb{R}^n$. The latter problem can easily be resolved by dropping the restriction $a_0 = 0$ and translating in the $(k+1)$st coordinate so that the barycenter $b = \sum_i w_i a_i$ is at 0.

To fix the other problem we choose "upper and lower barycenters" b^+ and b^- by letting $a_i^+ = \max\{a_i, b\}$ and $a_i^- = \min\{a_i, b\}$, then setting $b^+ = \sum_i w_i a_i^+$ and $b^- = \sum_i w_i a_i^-$. Then we have $a_0 \le b^- \le b \le b^+ \le a_p$, with all these inequalities strict unless they are all equalities (and hence C is the empty configuration). Now we define the map σ by taking the part of C in the open slab $S(b^-, b^+) = \mathbb{R}^k \times (b^-, b^+) \times \mathbb{R}^{n-k-1}$ and stretching this slab out to $\mathbb{R}^n$ by stretching the interval (b^-, b^+) to $(-\infty, +\infty)$ in the $(k+1)$st coordinate. More precisely, let $L: \mathbb{R}^n \to \mathbb{R}^n$ be the map that is the identity in all coordinates except the $(k+1)$st coordinate, where it is the composition of a fixed identification of $(-\infty, +\infty)$ with $(-1, +1)$ followed by the unique affine linear map taking $[-1, +1]$ onto $[b^-, b^+]$. Then σ of the given point in $B\mathcal{M}^{n,k}$ is defined to be $L^{-1}(C)$. Note that this definition works even in the degenerate case that $b^+ = b^- = 0$ when we are just stretching the empty configuration. The map σ is continuous since it is continuous on each product $\Delta^p \times (\mathcal{M}^{n,k})^p$ and is consistently defined when weights go to 0.

Now we show that σ induces a surjection $\pi_q(B\mathcal{M}^{n,k}) \to \pi_q(\mathcal{C}^{n,k+1})$. Consider a map $f: D^q \to \mathcal{C}^{n,k+1}$ taking ∂D^q to the basepoint of $\mathcal{C}^{n,k+1}$, the empty configuration. Thus we have configurations $C_t = f(t) \in \mathcal{C}^{n,k+1}$ for $t \in D^q$. For fixed $t \in D^q$ the configuration C_t projects to $\mathbb{R}^{k+1}$ to a discrete set, so we can choose a closed ball $B \subset \mathbb{R}^k$ and a closed interval $J \subset \mathbb{R}$ such that $B \times J$ is disjoint from the projection of C_t to $\mathbb{R}^{k+1}$. (When $k = 0$ the factor B is omitted.) By expanding each ball $B \times \{a\}$ for $a \in J$ to $\mathbb{R}^k \times \{a\}$, damping this expansion down to zero near ∂J, we obtain a deformation of C_t to a configuration disjoint from the slices $S(a) = \mathbb{R}^k \times \{a\} \times (0,1)^{n-k-1}$ for a in a subinterval of J. For nearby values of t we can use the same product $B \times J$ since this is still disjoint from C_t. Using compactness of D^q we can then choose a cover of D^q by finitely many open sets V_i with corresponding products $B_i \times J_i$ disjoint from C_t for $t \in V_i$. After shrinking the intervals J_i appropriately, we can assume they are all disjoint. Then we can perform the deformations of C_t for different products $B_i \times J_i$ independently, damping the deformation for $B_i \times J_i$ down to zero outside slightly smaller open sets U_i that still cover D^q. Performing these damped down deformations for all the products $B_i \times J_i$ produces a new map $f: D^q \to \mathcal{C}^{n,k+1}$ homotopic to the old one by a homotopy which is constant for $t \in \partial D^q$ since the configuration C_t is empty there. We discard the original f and work now with the new family $f(t) = C_t$.

We can choose slices $S(a_i)$ disjoint from C_t over U_i by choosing any point a_i in J_i, and we can choose corresponding weights w_i via a partition of unity subordinate to the cover by the sets U_i. Over a neighborhood of ∂D^q we can choose just the slice $S(0)$ with weight 1. By taking the parts of C_t in the slabs between adjacent slices $S(a_i)$ we obtain a map $g: D^q \to B\mathcal{M}^{n,k}$ taking ∂D^q to the basepoint. The composition σg is homotopic to f by expanding the intervals (b_t^-, b_t^+) for the family $g(t)$ to $(-\infty, +\infty)$, thereby deforming the maps L_t to the identity.

Showing σ is injective on π_q is done similarly. Here we start with a map $g: S^q \to B\mathcal{M}^{n,k}$ and a map $f: D^{q+1} \to \mathcal{C}^{n,k+1}$ restricting to σg on S^q. We deform the family $f(t) = C_t$ by the same procedure as before, expanding in various slices $S(a)$ to make C_t disjoint from slices $S(a_i)$ over sets U_i covering D^{q+1}, with weights w_i coming from a partition of unity supported in the sets U_i. Over S^q this deformation of f induces a corresponding deformation of g since slices are preserved during the deformation. We keep the same notations for the new f and g. We extend the maps L_t that occur in the definition of σg over S^q so that they are defined over all of D^{q+1} by deforming them to the identity as t moves across a collar neighborhood of S^q in D^{q+1}, then taking them to be the identity over the rest of D^{q+1}. We then obtain a map $f': D^{q+1} \to B\mathcal{M}^{n,k}$ by taking the parts of $L_t(C_t)$ between the slices $L_t(S(a_i))$, with the weights w_i on these slices. A homotopy from f to f' on S^q is obtained by letting the weights on the slices for f decrease to 0 while the weights on the slices $L_t(S(a_i))$ for f' increase from 0 to their chosen values w_i. Thus f is homotopic to a map that extends over D^{q+1}, which shows that σ is injective on π_q. $\square$

Proof of Proposition 1.2. For every topological monoid $\mathcal{M}$ there is a canonical map $\mathcal{M} \to \Omega B\mathcal{M}$ since $\mathcal{M}$ is the space of edges in $B\mathcal{M}$ and every edge is a loop, as $B\mathcal{M}$ has just one vertex when $\mathcal{M}$ is a monoid. It is a classical result that the map $\mathcal{M} \to \Omega B\mathcal{M}$ is a homotopy equivalence when $\mathcal{M}$ is connected, or more generally when $\pi_0\mathcal{M}$ is not just a monoid but in fact a group with respect to the multiplication induced by the monoid structure on $\mathcal{M}$.

When $\pi_0\mathcal{M}$ is not a group, the map $\mathcal{M} \to \Omega B\mathcal{M}$ cannot be a homotopy equivalence since $\pi_0\Omega B\mathcal{M}$ is always a group, namely $\pi_1 B\mathcal{M}$. The Group Completion Theorem can often be applied in this more general setting, however. For simplicity, suppose that $\pi_0\mathcal{M}$ is the monoid $\mathbb{Z}_{\geq 0}$ of nonnegative integers under addition, as will be the case in our main applications. The path-components $\mathcal{M}_p$ of $\mathcal{M}$ then correspond to integers $p \geq 0$ and there is a stabilization map $\mathcal{M}_p \to \mathcal{M}_{p+1}$ given by taking the product (on the right, say) with any element of $\mathcal{M}_1$. In our applications this map will be an injection $\mathcal{M}_p \hookrightarrow \mathcal{M}_{p+1}$ so one can form the direct limit of the $\mathcal{M}_p$'s as $\cup_p \mathcal{M}_p$, with the direct limit topology. The Group Completion Theorem then says that with one additional technical hypothesis there is an isomorphism

$$H_*(\mathbb{Z} \times \cup_p \mathcal{M}_p) \cong H_*(\Omega B\mathcal{M})$$

The technical hypothesis is that the left and right actions of $\pi_0\mathcal{M}$ on $H_*\mathcal{M}$ induced by left and right multiplication in $\mathcal{M}$ coincide. The reason for the name "Group Completion Theorem" can be seen by looking at what it says about H_0, or equivalently π_0. On the left side we have $\pi_0(\mathbb{Z}\times\cup_p\mathcal{M}_p)=\mathbb{Z}$ since $\cup_p\mathcal{M}_p$ is path-connected, as each $\mathcal{M}_p$ is path-connected. Thus passing from $\mathcal{M}$ to $\mathbb{Z}\times\cup_p\mathcal{M}_p$ has the effect on π_0 of converting $\mathbb{Z}_{\geq 0}$ to its group completion $\mathbb{Z}$. By the theorem, this is also what is happening to π_0 when we replace $\mathcal{M}$ by $\Omega B\mathcal{M}$.

In the case at hand we have $\mathcal{M}=\mathcal{M}^{n,k}$. The map $\mathcal{M}^{n,k}\longrightarrow\Omega B\mathcal{M}^{n,k}$ is part of a composition

$$\mathcal{C}^{n,k}\hookrightarrow\mathcal{M}^{n,k}\longrightarrow\Omega B\mathcal{M}^{n,k}\longrightarrow\Omega\mathcal{C}^{n,k+1}$$

where the last map is $\Omega\sigma$, a homotopy equivalence, as is the first map. From the definition of σ it is clear that the composition is homotopic to the map $\mathcal{C}^{n,k}\to\Omega\mathcal{C}^{n,k+1}$ translating a configuration from $-\infty$ to $+\infty$ in the $(k+1)$st coordinate. When $k>0$ we have $\pi_0\mathcal{M}^{n,k}=0$ since any configuration in $\mathcal{C}^{n,k}$ can be translated to ∞ in the kth coordinate, or alternatively, pushed to ∞ by a suitably chosen radial expansion in the $\mathbb{R}^k$ factor of $\mathbb{R}^n$. Hence the composition above is a homotopy equivalence when $k>0$.

For the case $k=0$ we have $\pi_0\mathcal{M}^{n,0}=\mathbb{Z}_{\geq 0}$, which is not a group, so the best we can do is apply the Group Completion Theorem to get an isomorphism $H_*(\mathbb{Z}\times\cup_p\mathcal{M}_p^{n,0})\cong H_*(\Omega B\mathcal{M}^{n,0})$. The technical hypothesis that the left and right actions of $\pi_0\mathcal{M}^{n,0}$ on $H_*(\mathcal{M}^{n,0})$ are the same follows from the fact that the multiplication in $\mathcal{M}^{n,0}$ is homotopy-commutative when $n\geq 2$ by an argument like the one showing that higher homotopy groups are abelian. Namely, given configurations C_1 and C_2 one can squeeze their second coordinates into the intervals $(0,1/2)$ and $(1/2,1)$, respectively, then commute them in the first coordinate, then expand their second coordinates back to $(0,1)$. When $n=1$ the multiplication is also homotopy-commutative since the components of $\mathcal{M}^{1,0}$ are contractible, so $\mathcal{M}^{1,0}$ is homotopy equivalent to $\mathbb{Z}_{\geq 0}$ which is commutative. $\square$

Some Variants

(1) Before letting n go to infinity in the preceding proof we had homology isomorphisms $H_*(\lim_p\mathcal{C}_p(\mathbb{R}^n))\cong H_*(\Omega_0^nS^n)$. When $n=2$ the space $\mathcal{C}_p(\mathbb{R}^2)$ is a classifying space for the braid group B_p so we obtain an isomorphism

$$H_*(\cup_pB_p)\cong H_*(\Omega_0^2S^2)$$

(2) Given a path-connected space X, we can enhance the space $\mathcal{C}^n$ to a space $\mathcal{C}^n(X)$ consisting of configurations of points in $\mathbb{R}^n$ labeled by points of X. All the earlier constructions extend immediately to this context. (When configurations are deformed at

various points in the proof, their labels are dragged along unchanged.) The generalization of Proposition 1.1 is that $\mathcal{C}^n(X)$ has the homotopy type of the subspace of labeled configurations with at most one point. This subspace is the quotient of $S^n \times X$ with $\{\infty\} \times X$ collapsed to a point. The usual way of writing this quotient is as $S^n(X_+)$, the n-fold reduced suspension of X_+, the space X with a disjoint basepoint attached. In other words, this is the smash product of S^n with X_+. Thus we obtain an isomorphism

$$H_*(\cup_p \mathcal{C}_p^n(X)) \cong H_*(\Omega_0^n S^n(X_+))$$

for each $n \geq 2$ including the limiting case $n = \infty$.

As a special case, let $X = BG$ for a discrete group G. Then $\mathcal{C}_p^\infty(BG)$ is a $B(G \wr \Sigma_p)$, because the projection $\mathcal{C}_p^\infty(BG) \to \mathcal{C}^\infty$ forgetting the labels is a fiber bundle with fiber $(BG)^p$, and it has a section so it is easy to see that $\pi_1 \mathcal{C}_p^\infty(BG)$ is the wreath product $G \wr \Sigma_p$. These observations lead to an isomorphism

$$H_*(\cup_p G \wr \Sigma_p) \cong H_*(\Omega_0^\infty S^\infty(BG_+))$$

The fact that the homology of $G \wr \Sigma_p$ stabilizes with increasing p is a classical result, and more recent proofs can be found in [HW], Proposition 1.6, and as a special case of Theorem 5 in [Han]. The homology isomorphism above can be found in [Sa].

2. The Madsen-Weiss Theorem

The Madsen-Weiss theorem can be proved following the same plan, but there are some points where additional arguments are needed. To emphasize the similarities we will use the same or very similar notation. Parts of the proof work for manifolds of arbitrary dimension, not just surfaces, so we will begin in this generality.

Let $\mathcal{C}(M, \mathbb{R}^\infty)$ be the space of oriented smooth submanifolds of $\mathbb{R}^\infty$ diffeomorphic to a given closed orientable manifold M of dimension m. By definition, $\mathcal{C}(M, \mathbb{R}^\infty)$ is the union of its subspaces $\mathcal{C}(M, \mathbb{R}^n)$ of oriented smooth submanifolds of $\mathbb{R}^n$ diffeomorphic to M, with the direct limit topology. As explained in the Appendix, $\mathcal{C}(M, \mathbb{R}^\infty)$ is a classifying space for smooth oriented fiber bundles with fiber M, but for the purposes of the Madsen-Weiss theorem the key feature of $\mathcal{C}(M, \mathbb{R}^\infty)$ is that it is a $K(\Gamma, 1)$ for Γ the mapping class group of M when $\mathrm{Diff}^+(M)$ has contractible path-components. Actually what will be needed is a relative version of this statement for manifolds with boundary. We will discuss this when it is needed, near the end of the proof.

We enlarge $\mathcal{C}(M, \mathbb{R}^n)$ to the space $\mathcal{C}^n$ of all smooth oriented m-dimensional submanifolds of $\mathbb{R}^n$ that are not necessarily closed, but are properly embedded, so they are closed subspaces of $\mathbb{R}^n$ even if they are not closed manifolds. The manifolds in $\mathcal{C}^n$ need not be connected, and the empty manifold is allowed and will be important, serving as a basepoint. The topology on $\mathcal{C}^n$ is defined to have as basis the sets $C(B,V)$ where:

(i) B is a closed ball in $\mathbb{R}^n$ centered at the origin.

(ii) V is an open set in the standard C^∞ topology on the space of m-dimensional smooth compact submanifolds of B that are properly embedded (meaning that the intersection of the submanifold with ∂B is the boundary of the submanifold, and this is a transverse intersection).

(iii) $C(B,V)$ consists of all manifolds M in $\mathcal{C}^n$ meeting ∂B transversely such that $M\cap B$ is in V.

The empty submanifold of B is allowed in (ii), and forms an open set V all by itself. It is not hard to check that the sets $C(B,V)$, as B and V vary, satisfy the conditions for defining a basis for a topology. In this topology a neighborhood basis for a given manifold $M \in \mathcal{C}^n$ consists of the subsets of $\mathcal{C}^n$ obtained by choosing a ball B with ∂B transverse to M and taking all manifolds $M' \in \mathcal{C}^n$ such that $M' \cap B$ is C^∞-close to $M \cap B$, so in particular M' is also transverse to ∂B.

This topology on $\mathcal{C}^n$ allows manifolds to be pushed to infinity by radial expansion from any point in the complement of a given manifold, so $\mathcal{C}^n$ is path-connected if $n > m$. If $n = m$, $\mathcal{C}^n$ consists of two points, the empty manifold and $\mathbb{R}^n$ itself. This case is not of interest for us, so *we will assume $n > m$ from now on without further mention.*

For the proof of the Madsen-Weiss theorem, when we take $m = 2$, it would be possible to restrict surfaces in $\mathcal{C}^n$ to have finite total genus over all components since the constructions we make will preserve this property, but there is no special advantage to adding this restriction.

The space $\mathcal{C}^n$ has a homotopy type that can be described fairly easily:

Proposition 2.1. *$\mathcal{C}^n$ has the homotopy type of its subspace $AG^+_{n,m}$ consisting of oriented affine linear m-planes in $\mathbb{R}^n$, together with the empty set which gives the one-point compactification of the space of m-planes.*

Proof. Consider the operation of rescaling $\mathbb{R}^n$ from the origin by a multiplicative factor $\lambda \geq 1$, including the limiting value $\lambda = \infty$. Applied to any manifold M defining a point in $\mathcal{C}^n$ this rescaling operation defines a path in $\mathcal{C}^n$ ending with an m-plane through the origin if $0 \in M$, or with the empty set if $0 \notin M$. This path does not depend continuously on M, however, since as we perturb a manifold containing 0 to one which does not, the endpoint of the path suddenly changes from a plane through the origin to the empty set. To correct for this problem we will modify the rescaling operation.

Let V be a tubular neighborhood of a given $M \in \mathcal{C}^n$, so V is a vector bundle $p: V \to M$ with fibers orthogonal to M. If $0 \notin V$ then we do just what we did in the previous paragraph, rescaling $\mathbb{R}^n$ from the origin by factors ranging from 1 to ∞. If $0 \in V$ then we rescale by different factors in the directions of the tangent and normal planes to M at $p(0)$. In the tangential direction we again rescale by a factor going from 1 to ∞, but in the normal direction we rescale by a factor ranging from 1 to a number λ that decreases from ∞ to 1 depending on the position of 0 in the fiber of V, with λ being ∞ when 0 is near the frontier of V and 1 when 0 is near the zero-section M in V, so there is no normal rescaling at all near M. This gives a continuous deformation of $\mathcal{C}^n$ into the subspace $AG^+_{n,m}$ consisting of linear planes and the empty set, using the fact that the tubular neighborhood V can be chosen to vary continously with M. The subspace $AG^+_{n,m}$ is taken to itself during the deformation, so the inclusion $AG^+_{n,m} \hookrightarrow \mathcal{C}^n$ is a homotopy equivalence. $\square$

We filter $\mathcal{C}^n$ by subspaces $\mathcal{C}^{n,0} \subset \mathcal{C}^{n,1} \subset \cdots \subset \mathcal{C}^{n,n} = \mathcal{C}^n$ where $\mathcal{C}^{n,k}$ is the subspace of $\mathcal{C}^n$ consisting of manifolds that are properly embedded in $\mathbb{R}^n$ but actually lie in the subspace $\mathbb{R}^k \times (0,1)^{n-k}$. Thus a manifold M in $\mathcal{C}^{n,k}$ can extend to infinity in only k directions, and the projection $M \to \mathbb{R}^k$ is a proper map, meaning that the inverse image of each compact set in $\mathbb{R}^k$ is compact in M.

The monoid version of $\mathcal{C}^{n,k}$ for $k < n$ is the subspace $\mathcal{M}^{n,k}$ of $\mathcal{C}^n \times [0,\infty)$ consisting of pairs (M,a) with M contained in $\mathbb{R}^k \times (0,a) \times (0,1)^{n-k-1}$. The inclusion $\mathcal{C}^{n,k} \hookrightarrow \mathcal{M}^{n,k}$, $M \mapsto (M,1)$, is a homotopy equivalence. The monoid structure in $\mathcal{M}^{n,k}$ induces a monoid structure in $\pi_0\mathcal{M}^{n,k} = \pi_0\mathcal{C}^{n,k}$. When $k = 0$ there are no inverses in this monoid structure on $\pi_0\mathcal{C}^{n,0}$ since the elements of this monoid are isotopy classes of closed oriented m-manifolds in $\mathbb{R}^n$. When $k > 0$, however, $\pi_0\mathcal{C}^{n,k}$ is a group, and we will need this fact so let us prove it now.

Proposition 2.2. *$\pi_0\mathcal{C}^{n,k} = 0$ when $k > m$, while for $0 < k \le m$ $\pi_0\mathcal{C}^{n,k}$ is isomorphic to the group $\Omega^{SO}_{m-k,n-k}$ of cobordism classes of closed oriented $(m-k)$-manifolds in $\mathbb{R}^{n-k}$, where cobordisms are embedded in $\mathbb{R}^{n-k} \times I$.*

Proof. A point in $\mathcal{C}^{n,k}$ is an m-dimensional manifold M in $\mathbb{R}^n$. As noted earlier, the projection $p: M \to \mathbb{R}^k$ is a proper map. By a small perturbation of M we can arrange that p is transverse to $0 \in \mathbb{R}^k$. If $k > m$ this implies that $p^{-1}(0)$ is empty. Then there is a ball B about 0 disjoint from $p(M)$ since p is proper. Expanding the $\mathbb{R}^k$ factor of $\mathbb{R}^n$ radially from 0 until B covers all of $\mathbb{R}^k$ then gives a path in $\mathcal{C}^{n,k}$ from M to the empty manifold, so $\pi_0\mathcal{C}^{n,k} = 0$ in the cases $k > m$.

When $k \leq m$, the submanifold $M_0 = p^{-1}(0) = M \cap (\{0\} \times \mathbb{R}^{n-k})$ of M inherits an orientation from the given orientation of M, so M_0 gives an element of $\Omega^{SO}_{m-k,n-k}$. By a standard transversality argument, the association $M \mapsto M_0$ determines a well-defined homomorphism $\varphi : \pi_0 \mathcal{C}^{n,k} \to \Omega^{SO}_{m-k,n-k}$. This is surjective since we can choose $M = \mathbb{R}^k \times M_0$ for any cobordism class $[M_0] \in \Omega^{SO}_{m-k,n-k}$.

To show that φ is injective, first deform a given M near M_0 so that M agrees with $\mathbb{R}^k \times M_0$ in a small neighborhood of M_0. Then by radial expansion of $\mathbb{R}^k$ we can construct a path in $\mathcal{C}^{n,k}$ connecting M to $\mathbb{R}^k \times M_0$, so we may assume $M = \mathbb{R}^k \times M_0$. If we are given another M' with $[M_0] = [M_0']$ in $\Omega^{SO}_{m-k,n-k}$, we may similarly assume $M' = \mathbb{R}^k \times M_0'$. Since $[M_0] = [M_0']$ there is a cobordism $V \subset I \times \mathbb{R}^{n-k}$ from M_0 to M_0'. Let W be the submanifold $\mathbb{R}^{k-1} \times V \subset \mathbb{R}^n$, extended by M in $\mathbb{R}^{k-1} \times (-\infty, 0] \times \mathbb{R}^{n-k}$ and by M' in $\mathbb{R}^{k-1} \times [1, \infty) \times \mathbb{R}^{n-k}$. Translating W from $+\infty$ to $-\infty$ in the kth coordinate of $\mathbb{R}^n$ then gives a path in $\mathcal{C}^{n,k}$ from M to M'. This shows φ is injective. □

For the next proposition we let $\mathcal{C}^{n,k+1}_0$ denote the component of $\mathcal{C}^{n,k+1}$ containing the empty manifold.

Proposition 2.3. $\mathcal{C}^{n,k+1}_0 \simeq B\mathcal{M}^{n,k}$ *for* $k > 0$. *Hence* $\mathcal{C}^{n,k} \simeq \Omega \mathcal{C}^{n,k+1}$ *for* $k > 0$.

Proof. This follows the plan of the proof of Proposition 1.3 quite closely. There is a map $\sigma : B\mathcal{M}^{n,k} \to \mathcal{C}^{n,k+1}$ defined as before. This has image in $\mathcal{C}^{n,k+1}_0$ since the image contains the empty manifold and $B\mathcal{M}^{n,k}$ is path-connected. One then argues as before that σ induces isomorphisms on all homotopy groups. The only difference in the argument comes in the process of deforming a given manifold M in $\mathcal{C}^{n,k+1}_0$ to be disjoint from a slice $S(a)$. The projection $p : M \to \mathbb{R}^{k+1}$ is a proper map, so its set of regular values is open and dense in $\mathbb{R}^{k+1}$. Let x be a regular value and let $M_x = p^{-1}(x)$, a manifold of dimension $m-k-1$. When $m < k+1$, M_x is empty, and the set of such values x is open and dense in $\mathbb{R}^{k+1}$, so in this case the proof can proceed exactly as in Proposition 1.3.

When $m \geq k+1$ the manifold M_x can be nonempty, and a slightly more complicated deformation is needed to make M disjoint from slices near the slice $S(a)$ containing x. This uses a process somewhat like the one used in the proof of Proposition 2.2. First perturb M near M_x to agree with $\mathbb{R}^{k+1} \times M_x$ in a neighborhood of M_x. Next, do the damped radial expansion in slices near $S(a)$ as in Proposition 1.3 to make M agree with $\mathbb{R}^{k+1} \times M_x$ in all slices in a neighborhood of $S(a)$. Since M is in the component $\mathcal{C}^{n,k+1}_0$ of $\mathcal{C}^{n,k+1}$, the cobordism class $[M_x]$ is zero in $\Omega^{SO}_{m-k-1,n-k-1}$, so M_x bounds a manifold V in $I \times \mathbb{R}^{n-k-1}$. We place the I factor of this product in the kth coordinate of $\mathbb{R}^k$. Then translation in this coordinate as in the proof of Proposition 2.2, but now damped down away from the slice $S(a)$, gives a further deformation of M to make it disjoint from this slice, the deformation being supported in a neighborhood of this slice. With this enhancement, the rest of the proof that $\sigma : B\mathcal{M}^{n,k} \to \mathcal{C}_0^{n,k+1}$ is a homotopy equivalence goes through as before.

The second statement of the proposition is obtained by combining four equivalences:

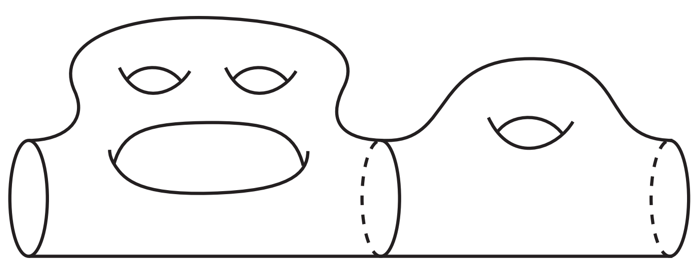
$$\Omega\mathcal{C}^{n,k+1} = \Omega\mathcal{C}_0^{n,k+1} \simeq \Omega B\mathcal{M}^{n,k} \simeq \mathcal{M}^{n,k} \simeq \mathcal{C}^{n,k}$$

The first equality is obvious since points in ΩX are loops in the basepoint path-component of X. The next equivalence $\Omega\mathcal{C}_0^{n,k+1} \simeq \Omega B\mathcal{M}^{n,k}$ holds since $\mathcal{C}_0^{n,k+1} \simeq B\mathcal{M}^{n,k}$ as we just proved. The third equivalence $\Omega B\mathcal{M}^{n,k} \simeq \mathcal{M}^{n,k}$ holds since $\pi_0\mathcal{M}^{n,k} \cong \pi_0\mathcal{C}^{n,k}$ is a group when $k > 0$, by Proposition 2.2. The last equivalence $\mathcal{M}^{n,k} \simeq \mathcal{C}^{n,k}$ has already been noted. ☐

There remains the more delicate matter of comparing $\mathcal{C}^{n,1}$ with $B\mathcal{M}^{n,0}$. At this point we will be restricting to manifolds of dimension $m = 2$, so our manifolds are surfaces. With the definition of $\mathcal{M}^{n,0}$ that we have given, the product operation in $\mathcal{M}^{n,0}$, juxtaposition in the first coordinate of $\mathbb{R}^n$, is disjoint union of closed surfaces, whereas what one needs for the Madsen-Weiss theorem is a sum operation within the realm of connected surfaces. With this in mind, we will replace $\mathcal{M}^{n,0}$ with a different monoid $\mathcal{M}^n$ defined in the following way. Consider the cylinder $Z = \mathbb{R} \times C \subset \mathbb{R} \times (0,1)^{n-1}$ where C is a fixed circle in the first two coordinates of $(0,1)^{n-1}$. Choose a baseline in Z of the form $\mathbb{R} \times \{p\}$ for p a basepoint in the circle C. Then let $\mathcal{M}^n$ be the space of pairs (S, a) where S is a compact connected oriented surface properly embedded in $[0,a] \times (0,1)^{n-1} \subset \mathbb{R}^n$ such that:

(i) ∂S consists of the two circles $Z \cap S(0)$ and $Z \cap S(a)$, and S is tangent to Z to infinite order along these two circles. Moreover the orientation of S agrees with a fixed orientation of Z along these circles.

(ii) S contains the segment of the baseline of Z in the slab $S[0,a]$ between $S(0)$ and $S(a)$, and S is tangent to Z along this segment. (This tangency condition is not essential.)



We also allow the limiting case $a = 0$ when S degenerates to just a circle. The monoid structure in $\mathcal{M}^n$ is given by juxtaposition in the first coordinate of $\mathbb{R}^n$. The condition of tangency to infinite order in (i) guarantees that the monoid operation stays within the realm of smooth surfaces.

We will only need the case $n = \infty$, so we set $\mathcal{M}^\infty = \cup_n \mathcal{M}^n$ with the direct limit topology as usual. The path-components of $\mathcal{M}^\infty$ are the subspaces $\mathcal{M}^\infty_g$ consisting of surfaces of genus g. A map $\sigma : B\mathcal{M}^\infty \to \mathcal{C}^{\infty,1}$ is defined just as before.

Proposition 2.4. *The map $\sigma : B\mathcal{M}^\infty \to \mathcal{C}^{\infty,1}$ is a homotopy equivalence.*

Proof. This will be more complicated than the preceding proofs, so we break the argument up into two separate pieces by defining a space $\mathcal{C}^{\infty,1}_s$ with maps

$$B\mathcal{M}^\infty \xleftarrow{\rho} \mathcal{C}^{\infty,1}_s \xrightarrow{\tau} \mathcal{C}^{\infty,1}$$

such that the composition $\sigma\rho$ is homotopic to τ, and then we show that ρ and τ are homotopy equivalences, which implies that σ is also a homotopy equivalence.

A point in $\mathcal{C}^{\infty,1}_s$ consists of a surface S in $\mathcal{C}^{\infty,1}$ with numbers $a_0 \le \cdots \le a_p$, $p \ge 0$, and corresponding weights $w_i \ge 0$ summing to 1, such that:

- (i) S agrees with the cylinder Z in each slice $S(a_i)$, where it is tangent to Z to infinite order, with orientations agreeing.
- (ii) S contains the baseline $\mathbb{R} \times \{p\}$ and is tangent to Z along the baseline. (Again this tangency condition is not essential.)
- (iii) The intersection of S with each slab between adjacent slices $S(a_i)$ is connected.

There is a map $\rho : \mathcal{C}^{\infty,1}_s \to B\mathcal{M}^\infty$ obtained by restricting to the the parts of surfaces between adjacent slices, and there is a map $\tau : \mathcal{C}^{\infty,1}_s \to \mathcal{C}^{\infty,1}$ forgetting the extra data. The composition $\sigma\rho$ is homotopic to τ by expanding the intervals $[b^-, b^+]$ in the definition of σ to $[-\infty, +\infty]$.

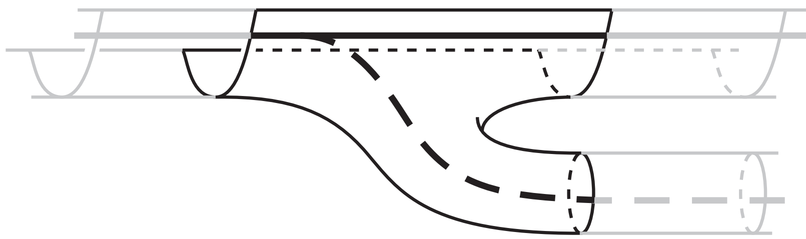
To show that ρ induces isomorphisms on all homotopy groups it suffices to show that for every map $f : D^q \to B\mathcal{M}^\infty$ with a lift $\widetilde{f} : \partial D^q \to \mathcal{C}^{\infty,1}_s$ there is a homotopy of f and a lifted homotopy of $\widetilde{f}$ to new maps g and $\widetilde{g}$ such that $\widetilde{g}$ extends to a lift of g over all of D^q, for arbitrary $q > 0$. (Both $B\mathcal{M}^\infty$ and $\mathcal{C}^{\infty,1}_s$ are path-connected so there is no need to consider the case $q = 0$.)

As in the definition of σ, we view f as defining a family of surfaces S_t in $\mathbb{R}^\infty$ with slices $S(a_i(t))$ and weights $w_i(t)$, for $t \in D^q$. The lift $\widetilde{f}$ over ∂D^q gives an enlargement of the surfaces S_t so that they extend to $\pm\infty$ in the first coordinate. We can assume the barycenters $b(t)$ are always at 0. The slices $S(a_i(t))$ cut S_t into pieces, and the first step in the deformation of f and $\widetilde{f}$ is to spread these pieces apart by inserting "padding" consisting of a piece of the cylinder Z of width $\varepsilon w_i(t)$ at the slice $S(a_i(t))$ for each i. This spreads the slices apart, and we take the new $S(a_i(t))$ to be at the center of the inserted piece of Z. Again we translate so that the new barycenters are at 0. Letting ε go from 0 to 1 gives a deformation of f and $\widetilde{f}$ to new maps f and $\widetilde{f}$ which we use for the next step of the argument.

We replace the construction of the intervals (b^-, b^+) by a different construction of intervals $(c^-(t), c^+(t))$ that contain at least one point $a_i(t)$ as follows. For fixed $t \in D^q$ choose slices $S(a^-(t))$ and $S(a^+(t))$ from among the slices $S(a_i(t))$ with nonzero weights $w_i(t)$, such that $a^-(t) \le 0 \le a^+(t)$. Expand the interval $[a^-(t), a^+(t)]$ slightly to an interval $[\varphi^-(t), \varphi^+(t)]$ by including a little of the padding around $a^-(t)$ and $a^+(t)$. The same choices of $a^\pm(t)$ and $\varphi^\pm(t)$ work in a small neighborhood U of t-values. Extend the functions $\varphi^\pm(t)$ over all of D^q by letting them be 0 outside a slightly larger neighborhood V and interpolating continuously in $V - U$. Since D^q is compact, finitely many such neighborhoods U suffice to cover D^q. Call these neighborhoods U_j, with corresponding intervals $[a_j^-(t), a_j^+(t)]$ and $[\varphi_j^-(t), \varphi_j^+(t)]$. Now define the interval $[c^-(t), c^+(t)]$ to be $\cup_j [\varphi_j^-(t), \varphi_j^+(t)]$. The key properties of $[c^-(t), c^+(t)]$ are that it is contained in a small neighborhood of $[a_0(t), a_p(t)]$ and it contains $[a_j^-(t), a_j^+(t)]$ for $t \in U_j$.

Let $g(t)$ be the family obtained from $f(t)$ by replacing the slices $S(a_i(t))$ with the slices $S(a_j^\pm(t))$, with weights $w_j^\pm(t)$ given by a partition of unity for the cover of D^q by the neighborhoods U_j. A homotopy from the family $f(t)$ to the family $g(t)$ is obtained by letting the weights go linearly from their values for $f(t)$ to their values for $g(t)$. Over ∂D^q this lifts to a homotopy of $\widetilde{f}(t)$ to a lift $\widetilde{g}(t)$ of $g(t)$ since we are not changing S_t. Finally, by expanding the intervals $(c^-(t), c^+(t))$ to $(-\infty, +\infty)$ we obtain homotopies of $g(t)$ and $\widetilde{g}(t)$ to $h(t)$ and $\widetilde{h}(t)$ such that $h(t)$ lifts to $\widetilde{h}(t)$ over all of D^q.

It remains to see that the map $\tau : \mathcal{C}_s^{\infty,1} \to \mathcal{C}^{\infty,1}$ is a homotopy equivalence. Define the subspace $\mathcal{C}_b^{\infty,1} \subset \mathcal{C}^{\infty,1}$ to consist of surfaces satisfying just the baseline condition (ii) above. We first show the inclusion $\mathcal{C}_b^{\infty,1} \hookrightarrow \mathcal{C}^{\infty,1}$ is an equivalence by showing the relative homotopy groups $\pi_q(\mathcal{C}^{\infty,1}, \mathcal{C}_b^{\infty,1})$ are zero. Let $f : (D^q, \partial D^q) \to (\mathcal{C}^{\infty,1}, \mathcal{C}_b^{\infty,1})$ represent an element of this homotopy group. We will modify the surfaces $S_t = f(t)$ in a neighborhood of the baseline so as to deform f into $\mathcal{C}_b^{\infty,1}$, staying in $\mathcal{C}_b^{\infty,1}$ over ∂D^q. The first step is to introduce a new component of S_t which is a thin copy of the cylinder Z close to Z. This can be done via a deformation in which half of this thin cylinder is capped off with a hemisphere giving a local maximum for projection onto the first coordinate of $\mathbb{R}^\infty$, and the deformation consists of translating this cap from $-\infty$ to $+\infty$. In $\mathbb{R}^\infty$ there is plenty of room to do this construction in the complement of S_t. Do this for all t simultaneously, and call the resulting family of enlarged surfaces S'_t. For $t \in \partial D^q$ we wish to merge the new cylinder in S'_t with a neighborhood of the baseline in S'_t, and we can do this by 'zipping' the two surfaces together, sliding the piece of a surface shown in the figure below from $-\infty$ to $+\infty$. (The baseline is the heavy solid horizontal line.)



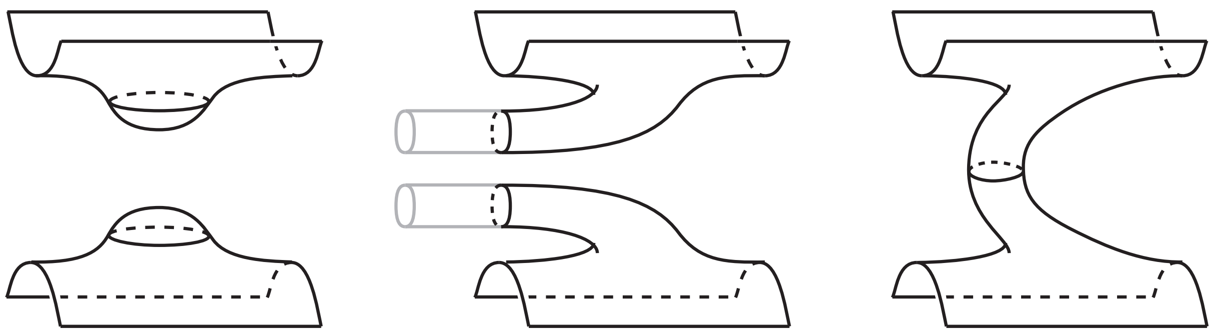
This zipping isotopy gives an extension of the family S'_t over an external collar $\partial D^q \times I$ attached to D^q, with the zipping operation being performed as the I coordinate in this collar increases from 0 to 1. Viewing the extended family S''_t still as a map $(D^q, \partial D^q) \to (\mathcal{C}^{\infty,1}, \mathcal{C}_b^{\infty,1})$, it is homotopic through such maps to the family S'_t by adding the collar gradually. The family S''_t agrees with the original S_t over ∂D^q. The baseline in S''_t for $t \in \partial D^q$ extends to a baseline over all of D^q by taking the extended baseline to lie in the new cylinder. This is indicated by the heavy dashed line in the figure. The extended baseline is not the standard baseline $\mathbb{R} \times \{p\}$, but it can be made to be the standard baseline by a slice-preserving ambient isotopy, giving a further deformation of S''_t. Thus we have deformed the given family S_t to lie in $\mathcal{C}_b^{\infty,1}$ over all of D^q, staying in $\mathcal{C}_b^{\infty,1}$ over ∂D^q during the deformation, so $\pi_q(\mathcal{C}^{\infty,1}, \mathcal{C}_b^{\infty,1}) = 0$.

To show that the map $\mathcal{C}_s^{\infty,1} \to \mathcal{C}_b^{\infty,1}$ induces an isomorphism on homotopy groups we start with $f : D^q \to \mathcal{C}_b^{\infty,1}$ and a partial lift $\widetilde{f} : \partial D^q \to \mathcal{C}_s^{\infty,1}$. There will be three steps. As notation we let $S[a,b]$ denote the slab between the slices $S(a)$ and $S(b)$.

(1) In this easy preliminary step we deform f and $\widetilde{f}$ so that the slices $S(a_i(t))$ in the family $\widetilde{f}$ all lie in the slab $S[-1,1]$. First we deform $\widetilde{f}$ so that the slices $S(a_i(t))$ vary continuously not just over the set where the corresponding weights $w_i(t)$ are nonzero, but over the closure of this set as well. We do this by modifying the weights by stretching the interval $[0, \max\{w_i(t)\}]$ to $[-\max\{w_i(t)\}, \max\{w_i(t)\}]$ and rescaling so that the new weights sum to 1. After doing this we do translations in the first coordinate of $\mathbb{R}^\infty$ to move the barycenter of the set of slices $S(a_i(t))$ to 0 for all $t \in \partial D^q$. Then we rescale the first coordinate to squeeze all these slices into $S[-1,1]$, using the fact that the values $a_i(t)$ are bounded as t ranges over the compact space ∂D^q. These translations and rescalings over ∂D^q can be extended over D^q, damping them off as we move into the interior of D^q.

(2) Next we deform the family S_t so that for each t, the subspace $S_t \cap S[-1,1]$ of S_t is contained in the baseline component of S_t. In other words, each point of $S_t \cap S[-1,1]$ can be joined to the baseline by a path in S_t, although this path need not stay in $S[-1,1]$. We

can achieve this as follows. For fixed t choose a slab $S[a,b]$ slightly larger than $S[-1,1]$ with $S(a)$ and $S(b)$ transverse to S_t. If a component C of $S_t \cap S[a,b]$ does not lie in the baseline component of S_t, we can create a tube in $S[a,b]$ joining C to the baseline component of S_t by the deformation shown in the figure below which takes a pair of small



disks in S_t, one in C and the other near the baseline, and drags these disks to $\pm\infty$ in the first coordinate of $\mathbb{R}^\infty$ to create a pair of tubes to $\pm\infty$, then these tubes are brought back from $\pm\infty$ joined together so that the given orientation of S_t extends over the new tube. We can do this simultaneously for each such component C, and after this is done, all of $S_t \cap S[-1,1]$ lies in the baseline component of S_t since $S_t \cap S[a,b]$ lies in the baseline component.

To do this procedure for all t we first use compactness of D^q to choose a finite cover of D^q by open sets U_i with slabs $S[a_i,b_i]$ containing $S[-1,1]$ such that the slices $S(a_i)$ and $S(b_i)$ are transverse to S_t over the closure $\overline{U}_i$ of U_i. The pairs of disks that we use to create tubes making $S_t \cap S[a_i,b_i]$ connected will be small neighborhoods of pairs of points p_{ij} and q_{ij}, with q_{ij} near the baseline. The main concern will be choosing all these points to be disjoint for fixed t and varying i and j. This is easy for the points q_{ij} which lie near the fixed baseline, so we focus attention on the points p_{ij}. We choose the p_{ij} by induction on i. For the induction step of choosing p_{ij} over $\overline{U}_i$, note first that the surface $S_t \cap S[a_i,b_i]$ varies by isotopy as t ranges over $\overline{U}_i$, so we can regard $S_t \cap S[a_i,b_i]$ as being independent of t over $\overline{U}_i$. Inductively we assume we have already chosen points p_{kl} for $k < i$, each point p_{kl} being defined and varying continuously over some closed ball in $\overline{U}_k$, with all these points being disjoint for each t. To begin the induction step we make an initial choice of points p_{ij} in the nonbaseline components of $S_t \cap S[a_i,b_i]$. To achieve disjointness from the previously chosen p_{kl} we will use a simple "scattering" trick to replace each p_{ij} by a finite number of nearby points, based on the following easy fact whose proof we leave as an exercise:

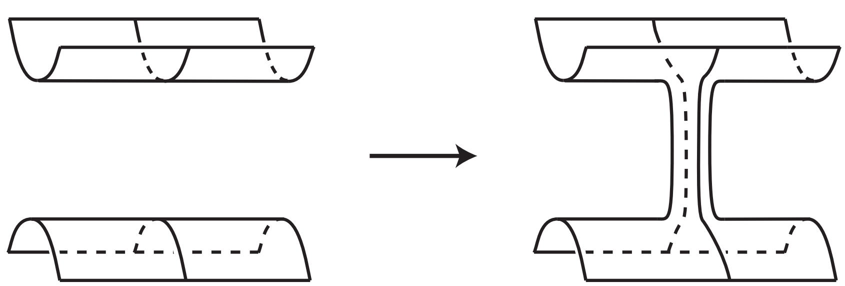
- Let K be a closed set in $D^p \times D^q$ intersecting each slice $D^p \times \{t\}$ in a finite set, with

$p > 0$. Then there exist finitely many distinct points $p_r \in D^p$ with corresponding open disks V_r covering D^q such that K is disjoint from $\{p_r\} \times \overline{V}_r$ for each r.

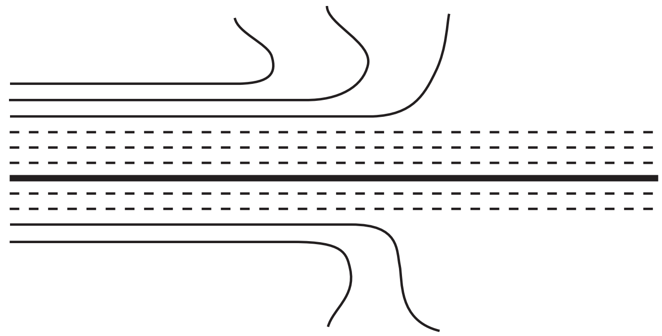
We apply this once for each point p_{ij}, with D^p a neighborhood of p_{ij} in S_t, $D^q = \overline{U}_i$, and K the union of the previously chosen points p_{kl} for t-values in $\overline{U}_i$. The result is that each p_{ij} is replaced by a number of points p_{ijr} with the desired disjointness. For convenience we relabel these points just as p_{ij}, each p_{ij} living over a subball $\overline{U}_{ij}$ of $\overline{U}_i$. This gives the induction step for choosing the points p_{ij}.

Having chosen all the pairs p_{ij}, q_{ij} to be disjoint for each t, we can deform the family S_t to create tubes joining these pairs, damping the deformation for the tube from p_{ij} to q_{ij} down to zero near the boundary of $\overline{U}_{ij}$, with this damping being done in such a way that the full isotopies take place in slightly shrunk versions of U_{ij} that still cover D^q. Since the ambient space is $\mathbb{R}^\infty$, we can choose all the tubes to be disjoint at all times. Over ∂D^q we can choose the tubes to lie outside the slices $S(a_i)$ and the slabs between these slices since the points of S_t in these slices and slabs all lie in the baseline component of S_t, and we arranged in step (1) that these slices and slabs lie inside $S[-1,1]$, so we can choose the tubes to go to either $+\infty$ or $-\infty$ so as to miss these slices and slabs. This guarantees that the deformation of f that we have constructed induces a corresponding deformation of $\widetilde{f}$ over ∂D^q.

(3) The final step is to make each S_t intersect at least one slice $S(a)$ in $S[-1,1]$ transversely in a connected set, a single circle. The idea will be to insert tubes that lie in a neighborhood of $S(a)$ so as to join different circles of $S_t \cap S(a)$ together, as shown in the figure below. Doing this for parametrized families will require a more refined variant of the procedure we used in the preceding step to connect different components of S_t together.



Suppose S_t meets a slice $S(a)$ in $S[-1,1]$ transversely in more than one circle. For each circle C_j of $S_t \cap S(a)$ that does not intersect the baseline choose a path α_j in S_t from a point $p_j \in C_j$ to a point q_j near the baseline. Step (2) guarantees that such a path exists. We can assume that α_j is an embedded arc. (It will not matter if α_j intersects some C_k's at interior points of α_j.) We extend the end of α_j near the baseline by a ray to $-\infty$ parallel to the baseline, still calling the extended arc α_j. In the figure below these are the thin solid arcs, with the baseline being the thick solid arc. We also insert corresponding disjoint arcs β_j parallel to the baseline and closer to it, extending to $+\infty$ as well as $-\infty$. These are shown dashed in the figure.



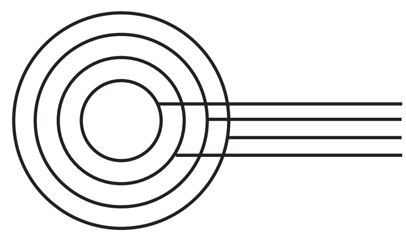
If the arcs α_j are in general position, they will intersect only in interior points that lie in the sections not parallel to the baseline, and then we can eliminate any intersections of α_j with other α_k's by pushing these α_k's off the end of α_j at p_j by an isotopy $S_t \to S_t$ supported near α_j. This does not introduce any intersections among the other α_k's since the isotopy of the α_k's is the restriction of an isotopy of S_t. Doing this for each α_j in turn, we can thus take all the α_j's (and β_j's) to be disjoint.

We can modify S_t by a surgery operation that attaches a thin tube T_j joining a point x_j of α_j to the point y_j of β_j having the same first coordinate as x_j. We embed T_j in $\mathbb{R}^\infty$ so that it lies in a small neighborhood of the slice through x_j and y_j. By letting x_j move along α_j from the end at $-\infty$ to the end at C_j while y_j moves simultaneously along β_j we obtain a deformation of S_t. Doing this for all the circles C_j at once, we deform S_t to a surface S'_t such that $S'_t \cap S(a)$ is connected. (There is no problem with making all the tubes T_j disjoint in $\mathbb{R}^\infty$.) By an isotopy of S'_t supported near $S(a)$ and fixed near the baseline we can then make S'_t coincide with the cylinder Z near $S(a)$.

For nearby t-values the curves C_j in $S_t \cap S(a)$ vary by small isotopies and we can take the arcs α_j and β_j to vary by small isotopies as well, staying disjoint for each t. By compactness of D^q we can then choose a cover of D^q by open balls U_i such that over $\overline{U}_i$ there is a slice $S(a_i)$ in $S[-1,1]$ transverse to S_t, with arcs α_{ij} going from $-\infty$ to the components C_{ij} of $S_t \cap S(a_i)$ disjoint from the baseline, along with corresponding arcs β_{ij} near the baseline. For fixed i all the arcs α_{ij} and β_{ij} are disjoint for each $t \in \overline{U}_i$, and our next task will be to make them disjoint as i varies as well.

The arcs β_{ij} are straight lines parallel to the baseline and can easily be chosen to be disjoint by induction on i. For the arcs α_{ij} we also proceed by induction on i, so we assume inductively that we have already made the arcs α_{kl} with $k < i$ disjoint for each t. We will show in the next paragraph how we can do a preliminary adjustment so that for each $t \in \overline{U}_i$, no arc α_{ij} passes through the endpoint p_{kl} of another α_{kl} with $k < i$. Assuming this has been done, any intersections of α_{ij} with arcs α_{kl} for $k < i$ will involve only interior points of α_{kl}. These intersections can be eliminated by the same sort of procedure as before, pushing the arcs α_{kl} off the end of α_{ij} at p_{ij} by an isotopy $S_t \to S_t$ supported near α_{ij}, damping this isotopy down to zero outside U_i.

The preliminary adjustment so that arcs α_{ij} are disjoint from the endpoints p_{kl} of arcs α_{kl} with $k < i$ can be done as follows by a variant of the scattering trick used in step (2) above. First thicken the arcs α_{ij} to narrow bands of arcs parallel to α_{ij}. Then for a fixed t all but finitely many of these parallel arcs will be disjoint from the finitely many endpoints p_{kl} with $k < i$. Choose one of these arcs disjoint from p_{kl}'s for each j. These choices extend continuously for nearby values of t as well. Thus we can cover $\overline{U}_i$ by finitely many neighborhoods U_{ir} with new choices of arcs α_{ijr} in each of these neighborhoods, such that α_{ijr} satisfies the disjointness condition we are trying to achieve. We can choose the arcs α_{ijr} to be disjoint from each other for each $t \in \overline{U}_i$ since we are free to choose α_{ijr} from an open set of the arcs parallel to α_{ij}. We choose corresponding slices $S(a_{ir})$ near $S(a_i)$ and connect α_{ijr} to the corresponding circle C_{ijr} of $S_t \cap S(a_{ir})$ as in the figure. To complete the induction step we then replace U_i, $S(a_i)$, C_{ij}, and α_{ij} by the collections U_{ir}, $S(a_{ir})$, C_{ijr}, and α_{ijr} and then relabel to eliminate the extra subscripts r.



Having all the arcs α_{ij} and β_{ij} disjoint, we can use these to construct a well-defined deformation of the family S_t to create tubes T_{ij} over U_i making S_t agree with the cylinder near $S(a_i)$ as described earlier, damping this deformation down near the boundary of U_i as usual. For the resulting family S'_t there is one problem remaining, however. Because the deformations for each U_i are damped off near the boundary of U_i, the tubes T_{ij} may intersect the slices $S(a_k)$ for other U_k's, perhaps destroying the connectedness of the intersections $S'_t \cap S(a_k)$. To avoid this problem we first make sure the tubes T_{ij} are very thin while they move to their final destination. Then there will always be plenty of slices $S(a_{kl})$ near $S(a_k)$ that are disjoint from the moving thin tubes, and we use these slices $S(a_{kl})$ instead of the original slices $S(a_k)$. (This is another instance of the scattering idea.)

Once we have deformed S_t to meet certain slices $S(a_k)$ in single circles, the parts of S_t in the slabs between these slices will be connected since inserting the tubes T_{ij} does not destroy the property that these parts of S_t lie in the baseline component of S_t, from step (2). Thus we have constructed a homotopy of the family $f(t) \in \mathcal{C}_b^{\infty,1}$ to a family $g(t)$ having a lift $\widetilde{g}(t) \in \mathcal{C}_s^{\infty,1}$ obtained by choosing weights for the slices $S(a_k)$. Over ∂D^q

the slices $S(a_i)$ of $\widetilde{f}(t)$ all lie in the slab $S[-1,1]$ by step (1), so in step (3) there is no need to modify S_t for $t \in \partial D^q$. ◻

The argument in the preceding proof can be applied to manifolds of higher dimension $m > 2$, but it proves a weaker result, namely that $\mathcal{C}^{\infty,1}$ is homotopy equivalent not to the classifying space of a monoid, but to the classifying space of a topological category, a category whose objects are smooth closed connected oriented manifolds of dimension $m-1$ embedded in $\mathbb{R}^\infty$ and whose morphisms are connected oriented cobordisms of dimension m embedded in slabs $[0,a] \times \mathbb{R}^\infty$.

The last remaining step in the proof of the Madsen-Weiss theorem is to relate $\mathcal{M}^\infty$ to the space $\mathcal{C}(S_\infty, \mathbb{R}^\infty)$ that appears in the statement of the theorem in the introduction. The component $\mathcal{M}_g^\infty$ of $\mathcal{M}^\infty$ consisting of genus g surfaces deformation retracts onto the subspace $\mathcal{M}_g^\infty(g)$ consisting of such surfaces in $[0,g] \times (0,1)^\infty$ when $g > 0$ (and also when $g = 0$, but we won't need this case). The space $\mathcal{C}(S_\infty, \mathbb{R}^\infty)$ is the union of its subspaces $\mathcal{C}_g(S_\infty, \mathbb{R}^\infty)$ consisting of properly embedded surfaces in $\mathbb{R}^\infty$ diffeomorphic to S_∞ and agreeing with a standard embedded S_∞ outside $S_{g,1}$, the unrestricted part the surface being diffeomorphic to $S_{g,1}$. For the standard S_∞ we take a surface in $\mathbb{R}^3 \subset \mathbb{R}^\infty$ that intersects the half-infinite slab $S(-\infty, g]$ in $S_{g,1}$ and intersects the slice $S(g)$ in the circle where the cylinder Z meets $S(g)$ and is tangent to Z to infinite order along this circle, for each $g \geq 0$. There is then an inclusion map $\mathcal{M}_g^\infty(g) \hookrightarrow \mathcal{C}_g(S_\infty, \mathbb{R}^\infty)$ obtained by adjoining the parts of the standard S_∞ outside the slab $S[0,g]$.

Proposition 2.5. *The inclusion* $\mathcal{M}_g^\infty(g) \hookrightarrow \mathcal{C}(S_{g,1}, \mathbb{R}^\infty)$ *is a homotopy equivalence.*

Proof. This will follow by considering a commutative diagram of fiber bundles:

$$\begin{array}{ccccc} \mathrm{Diff}'(S_{g,1}) & \longrightarrow & \mathcal{E}_g'(S_\infty, \mathbb{R}^\infty) & \longrightarrow & \mathcal{C}_g'(S_\infty, \mathbb{R}^\infty) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Diff}(S_{g,1}) & \longrightarrow & \mathcal{E}_g(S_\infty, \mathbb{R}^\infty) & \longrightarrow & \mathcal{C}_g(S_\infty, \mathbb{R}^\infty) \end{array}$$

In the second row $\mathcal{E}_g(S_\infty, \mathbb{R}^\infty)$ is the space of embeddings $S_\infty \to \mathbb{R}^\infty$ that agree with the standard embedding outside $S_{g,1}$, and the map to $\mathcal{C}_g(S_\infty, \mathbb{R}^\infty)$ takes the image of such an embedding. The spaces in the first row are subspaces of those in the second. The space $\mathcal{E}_g'(S_\infty, \mathbb{R}^\infty)$ consists of embeddings $f : S_\infty \to \mathbb{R}^\infty$ that agree with the standard embedding outside the slab $S[0,g]$ and along the part of the baseline in this slab, with tangent planes also standard along this line. The images of these embeddings form the subspace $\mathcal{C}_g'(S_\infty, \mathbb{R}^\infty)$ of $\mathcal{C}_g(S_\infty, \mathbb{R}^\infty)$, which is just the image of the inclusion map $\mathcal{M}_g^\infty(g) \hookrightarrow \mathcal{C}_g(S_\infty, \mathbb{R}^\infty)$. The fiber $\mathrm{Diff}'(S_{g,1}) \subset \mathrm{Diff}(S_{g,1})$ in the upper row consists of diffeomorphisms fixing a disk in the interior of $S_{g,1}$ and an arc joining this disk to $\partial S_{g,1}$, so the inclusion $\mathrm{Diff}'(S_{g,1}) \hookrightarrow \mathrm{Diff}(S_{g,1})$ is a homotopy equivalence. The fiber bundle property in the two rows can be checked by verifying the local triviality over neighborhoods in the base consisting of all sections of a normal bundle to a given subsurface $f(S_\infty)$. Projection to the fiber over $f(S_\infty)$ is given by composition of an embedding with the projection of the normal bundle onto its 0-section.

The total spaces in the two bundles are contractible. For $\mathcal{E}_g(S_\infty, \mathbb{R}^\infty)$ a contraction can be constructed in three stages. Regarding $\mathbb{R}^\infty$ as $(\mathbb{R}^3)^\infty$, define a linear isotopy $\mathbb{R}^\infty \to \mathbb{R}^\infty$ by the formula $g_t(v_0, v_1, v_2, \cdots) = (v_0, (1-t)v_1, (1-t)v_2 + tv_1, (1-t)v_3 + tv_2, \cdots)$ so that g_0 is the identity and g_1 shifts all coordinates after v_0 one unit to the right. From the formula it is easy to check that g_t is injective for all t, having trivial kernel. The first step in the contraction is to compose each embedding $f \in \mathcal{E}_g(S_\infty, \mathbb{R}^\infty)$ with the isotopy g_t. This yields embeddings whose v_1 coordinates are 0. Next we deform these embeddings by inserting a function th in the v_1 coordinate where $h : S_\infty \to \mathbb{R}^3$ is a smooth map that vanishes outside $S_{g,1}$ and is an embedding in the interior of $S_{g,1}$. The last step of the contraction takes an embedding f of the type we have produced so far and isotopes it to a fixed such embedding f_0 via the linear isotopy $(1-t)f + tf_0$. This produces embeddings for each t since the v_1 coordinate is h for each t, and the embeddings lie in $\mathcal{E}_g(S_\infty, \mathbb{R}^\infty)$ since the isotopy is stationary outside $S_{g,1}$. Thus we have a contraction of $\mathcal{E}_g(S_\infty, \mathbb{R}^\infty)$. A contraction of the subspace $\mathcal{E}'_g(S_\infty, \mathbb{R}^\infty)$ is constructed in the same way by choosing h to vanish also on the union of the disk and arc in $S_{g,1}$ where all the embeddings in $\mathcal{E}'_g(S_\infty, \mathbb{R}^\infty)$ are standard, with h an embedding where is is not required to vanish.

The result now follows by the five lemma applied to the long exact sequences of homotopy groups for the two bundles. □

Proof of the Madsen-Weiss Theorem. This is obtained by composing seven isomorphisms:

$$
\begin{aligned}
H_*(\mathcal{C}(S_\infty, \mathbb{R}^\infty)) &\cong H_*(\lim_g \mathcal{M}_g^\infty) && (1)\\
&\cong H_*(\Omega_0 B\mathcal{M}^\infty) && (2)\\
&\cong H_*(\Omega_0 \mathcal{C}^{\infty,1}) && (3)\\
&\cong \lim_n H_*(\Omega_0 \mathcal{C}^{n,1}) && (4)\\
&\cong \lim_n H_*(\Omega_0^n \mathcal{C}^n) && (5)\\
&\cong \lim_n H_*(\Omega_0^n AG_{n,2}^+) && (6)\\
&\cong H_*(\Omega_0^\infty AG_{\infty,2}^+) && (7)
\end{aligned}
$$

The first isomorphism follows from the preceding proposition by passing to limits as g goes to infinity.

The second isomorphism is an application of the Group Completion Theorem, once we check the condition that the left and right actions of $\pi_0\mathcal{M}^\infty$ on $H_*(\mathcal{M}^\infty)$ coincide. This can be seen by representing a homology class by a map from a finite Δ-complex to $\mathcal{M}^\infty$ and an element of π_0 by a standard embedding of a surface of given genus, then shrinking this surface to be small and lie near the baseline where it can easily be commuted with the surfaces representing the homology class. (With a little more effort one can show the product in $\mathcal{M}^\infty$ is homotopy-commutative.)

The third isomorphism uses Proposition 2.4 which implies that $\Omega_0 B\mathcal{M}^\infty \simeq \Omega_0\mathcal{C}^{\infty,1}$. The fourth isomorphism holds since homology and the loopspace functor commute with direct limit for a space that is the union of a sequence of subspaces with the direct limit topology. The fifth isomorphism follows by repeated applications of Proposition 2.3. The sixth is Proposition 2.1, and the seventh follows in the same way as the fourth. □

Some Variants

(1) The restriction to oriented surfaces can easily be dropped. The statement is then that for the standard nonorientable surface N_∞ of infinite genus there is an isomorphism

$$H_*(B\mathrm{Diff}_c(N_\infty)) \cong H_*(\Omega_0^\infty \overline{AG}^+_{\infty,2})$$

where $\overline{AG}$ is the version of AG without chosen orientations on the affine planes. Thus $\overline{AG}^+_{n,2}$ is the Thom space of a bundle over the Grassmann manifold of unoriented 2-planes in $\mathbb{R}^n$, namely the bundle of vectors normal to these 2-planes. In the nonorientable case the monoid $\pi_0\mathcal{M}^\infty$ is not $\mathbb{Z}_{\geq 0}$ but something slightly more complicated, corresponding to diffeomorphism classes of closed connected surfaces under connected sum. The group completion of this monoid is still $\mathbb{Z}$, however. The theorem can be restated in terms of mapping class groups of nonorientable surfaces since these satisfy homology stability by [W1] and the Earle-Eells theorem applies also to nonorientable surfaces.

(2) Surfaces with punctures can be treated in the same way, provided that one stabilizes with respect to both genus and number of punctures. Viewing the punctures as distinguished points on a surface rather than deleted points, let $\mathrm{Diff}_c(S_\infty, P)$ be the group of compactly supported diffeomorphisms of the infinite genus surface S_∞ that leave an infinite discrete closed set P in S_∞ invariant, perhaps permuting finitely many of the points of P. The space $AG_{n,2}$ is enlarged to a space $A^*G_{n,2}$ of oriented affine 2-planes in $\mathbb{R}^n$ with at most one distinguished point in the 2-plane, where the case of no distinguished point is regarded as the limiting case of a distinguished point that approaches infinity in the 2-plane. The statement of the theorem is that there is an isomorphism

$$H_*(B\mathrm{Diff}_c(S_\infty, P)) \cong H_*(\Omega_0^\infty A^*G^+_{\infty,2})$$

The space $AG_{n,2}$ is a retract of $A^*G_{n,2}$ by the map that forgets the distinguished point, and this retraction extends to a retraction of one-point compactifications. The quotient space $A^*G_{n,2}^+/AG_{n,2}^+$ can be identified with the Thom space of the trivial n-dimensional vector bundle over $G_{n,2}$ by regarding this bundle as the sum of the two canonical bundles over $G_{n,2}$ consisting of vectors in a given 2-plane and vectors orthogonal to it; the vector in the plane gives a distinguished point in the plane, and the vector orthogonal to it tells where to translate the plane. In the stable homotopy category retractions give wedge sum splittings, so we have an equivalence

$$\Omega_0^\infty A^*G_{\infty,2}^+ \simeq \Omega_0^\infty AG_{\infty,2}^+ \times \Omega_0^\infty S^\infty((G_{\infty,2})_+)$$

using the fact that the Thom space of a trivial n-dimensional bundle over X is the n-fold suspension $S^n(X_+)$, the subscript $+$ denoting union with a disjoint basepoint.

This theorem gives information about homology of mapping class groups of finite surfaces since $\mathrm{Diff}_c(S_\infty, P)$ has contractible components and homology stability is known to hold for mapping class groups not just for stabilization with respect to genus but also with respect to the number of punctures [BT], [Han], [HW].

More refined results that cover stabilization with respect to genus for a fixed number of punctures can be found in [BT].

Appendix: Classifying Spaces for Diffeomorphism Groups

The most classical of classifying spaces are the classifying spaces for vector bundles. The classifying space for real vector bundles of dimension k is the Grassmannian $G_{\infty,k}$ of k-dimensional vector subspaces of $\mathbb{R}^\infty$, the direct limit of the Grassmann manifolds $G_{n,k}$ of k-dimensional vector subspaces of $\mathbb{R}^n$. (This notation is not quite consistent with the notation earlier in the paper, where $G_{\infty,k}$ meant the 2-sheeted cover consisting of oriented k-dimensional vector subspaces of $\mathbb{R}^\infty$.) There is a canonical vector bundle $E_{\infty,k}$ over $G_{\infty,k}$ consisting of the pairs $(v,P) \in \mathbb{R}^\infty \times G_{\infty,k}$ with $v \in P$. This is a universal k-dimensional vector bundle in the sense that every vector bundle $E \to X$ is induced from the universal bundle by some map $X \to G_{\infty,k}$ which is unique up to homotopy. Some mild restrictions on X are needed. It suffices to assume for example that X is paracompact.

If we shift our point of view from vector spaces to automorphisms of vector spaces, there is an associated fiber bundle over $G_{\infty,k}$ whose fibers are $GL(k,\mathbb{R})$ rather than $\mathbb{R}^k$. This bundle is often written as $EGL(k,\mathbb{R}) \to BGL(k,\mathbb{R})$ where $BGL(k,\mathbb{R})$ is just another notation for $G_{\infty,k}$ and $EGL(k,\mathbb{R})$ is the space of linear embeddings $f:\mathbb{R}^k \to \mathbb{R}^\infty$, with the projection $EGL(k.\mathbb{R}) \to BGL(k,\mathbb{R})$ sending an embedding f to its image $f(\mathbb{R}^k)$. Since a linear embedding f is determined by where it sends the standard basis of $\mathbb{R}^k$, one could also describe $EGL(k,\mathbb{R})$ as the Stiefel manifold of k-tuples of linearly independent vectors in $\mathbb{R}^\infty$. The condition of linear independence can be strengthened to orthonormality, which amounts to requiring the embeddings f to be isometric embeddings, and then $GL(k,\mathbb{R})$ is replaced by the orthogonal group $O(k)$ and one has a fiber bundle $EO(k) \to BO(k)$ with fiber $O(k)$. This does not affect the homotopy types of the spaces, and in fact $BO(k)$ is the same space as $BGL(k,\mathbb{R})$, namely $G_{\infty,k}$.

A key feature of $EGL(k,\mathbb{R})$ and $EO(k)$ is that they are contractible. It is an elementary fact (see [H1], Proposition 4.66) that whenever one has a fiber bundle or fibration $F \to E \to B$ with E contractible, then there is a (weak) homotopy equivalence $F \to \Omega B$. Thus $O(k) \simeq \Omega BO(k) = \Omega G_{\infty,k}$.

Entirely analogous considerations hold for smooth fiber bundles with fiber a smooth compact manifold M. A classifying space for such bundles is the space $B\mathrm{Diff}(M) = \mathcal{C}(M,\mathbb{R}^\infty)$ of smooth submanifolds of $\mathbb{R}^\infty$ diffeomorphic to M. (The notation $\mathcal{C}(M,\mathbb{R}^\infty)$ conflicts with that used earlier in the paper since we are ignoring orientations now.) The topology on $\mathcal{C}(M,\mathbb{R}^\infty)$ is the direct limit of its subspaces $\mathcal{C}(M,\mathbb{R}^n)$ which are in turn given the quotient topology obtained by regarding $\mathcal{C}(M,\mathbb{R}^n)$ as the orbit space of the space of smooth embeddings $M \to \mathbb{R}^n$ (with the usual C^∞ topology) under the action of $\mathrm{Diff}(M)$ by composition.

There is a canonical bundle $\mathcal{E}(M,\mathbb{R}^\infty) \to \mathcal{C}(M,\mathbb{R}^\infty)$ with fiber M, where $\mathcal{E}(M,\mathbb{R}^\infty)$ consists of pairs (v,P) in $\mathbb{R}^\infty \times \mathcal{C}(M,\mathbb{R}^\infty)$ with $v \in P$. To see that this bundle is locally a product, consider a given $P \subset \mathbb{R}^n$ diffeomorphic to M with an open tubular neighborhood N that is identified with the normal bundle to P in $\mathbb{R}^n$. Then the sections of this bundle form a neighborhood of P in $\mathcal{C}(M,\mathbb{R}^n)$, and projecting these sections onto the 0-section P gives a local product structure for the projection $\mathcal{E}(M,\mathbb{R}^n) \to \mathcal{C}(M,\mathbb{R}^n)$, compatibly for increasing n, hence also for the direct limit $\mathcal{E}(M,\mathbb{R}^\infty) \to \mathcal{C}(M,\mathbb{R}^\infty)$.

The bundle $\mathcal{E}(M,\mathbb{R}^\infty) \to \mathcal{C}(M,\mathbb{R}^\infty)$ is universal for smooth bundles with fiber M over paracompact base spaces, by essentially the same argument that shows the vector bundle $E_{\infty,k} \to G_{\infty,k}$ is universal; see for example the first chapter of [H2]. Realizing a given bundle $M \to E \to B$ as a pullback of the universal bundle is equivalent to finding a map $E \to \mathbb{R}^\infty$ which is a smooth embedding on each fiber. Locally in the base B such maps exist by combining a local projection onto the fiber M with a fixed embedding $M \to \mathbb{R}^n$. These local maps to $\mathbb{R}^n$ that are embeddings on fibers can then be combined to a global map $E \to \mathbb{R}^\infty$ that is an embedding on fibers via a partition of unity argument just as in the vector bundle case. Thus every bundle $M \to E \to B$ is a pullback of the universal bundle via some map $B \to \mathcal{C}(M,\mathbb{R}^\infty)$. Uniqueness of this map up to homotopy follows from the uniqueness of the map $E \to \mathbb{R}^\infty$ up to homotopy, which is shown just as for vector bundles by composing with linear embeddings $\mathbb{R}^\infty \to \mathbb{R}^\infty$ onto the odd or even coordinates, using straight-line homotopies.

There is also an associated bundle $\mathrm{Diff}(M) \to E\mathrm{Diff}(M) \to B\mathrm{Diff}(M)$ whose total space $E\mathrm{Diff}(M)$ is the space of smooth embeddings $M \to \mathbb{R}^\infty$. The fiber bundle property can be proved using tubular neighborhoods and sections as before. The space $E\mathrm{Diff}(M)$ is contractible by pushing each embedding $M \to \mathbb{R}^\infty$ into the odd coordinates by composing with a linear isotopy of $\mathbb{R}^\infty$ into the odd coordinates, then taking a linear isotopy to a fixed embedding of M into the even coordinates.

Since we have a fiber bundle $\mathrm{Diff}(M) \to E\mathrm{Diff}(M) \to B\mathrm{Diff}(M)$ with contractible total space, it follows that $\mathrm{Diff}(M)$ is weakly equivalent to $\Omega B\mathrm{Diff}(M)$. In particular, $B\mathrm{Diff}(M)$ is a $K(\pi, 1)$ for the mapping class group of M if the components of $\mathrm{Diff}(M)$ are contractible.

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