Moduli spaces from a topological viewpoint

Ib Madsen

Abstract. This text aims to explain what topology, at present, has to say about a few of the many moduli spaces that are currently under study in mathematics.

The most prominent one is the moduli space $\mathcal{M}_g$ of all Riemann surfaces of genus g. Other examples include the Gromov–Witten moduli space of pseudo-holomorphic curves in a symplectic background, the moduli space of graphs and Waldhausen's algebraic K-theory of spaces.

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Introduction

The classical Riemann moduli space $\mathcal{M}_g$ is a $(6g-6)$-dimensional manifold with mild singularities (an orbifold). One would like to characterize its homotopy type, but in reality one must settle for less. Even the rational cohomology ring of $\mathcal{M}_g$ appears to be far too difficult; it is known today only for $g \leq 4$.

The central theme of the article revolves around Mumford's standard conjecture about the stable cohomology of $\mathcal{M}_g$, settled in my joint work with Michael Weiss a few years back [47]. The conjecture predicts the rational cohomology groups of $\mathcal{M}_g$ in a modest range of dimensions, the stable range. More accurately, [47] proves a generalized version of Mumford's conjecture, proposed in [46]: the (integral) cohomology ring of the (infinite genus) mapping class group is equal to the cohomology ring of a rather well-studied space in algebraic topology, a space associated with cobordism theory. From this Mumford's conjectured answer for the stable rational cohomology of $\mathcal{M}_g$ is easily deduced.

The new topological method in the study of the Riemann moduli space, presented below, has three key ingredients: Harer's stability theorem resulting from the action of mapping class groups on complexes of curves, Phillips' submersion theorem in singularity theory and Gromov's generalization thereof, and not least the Pontryagin–Thom theory of cobordisms of smooth manifolds. These tools are all rather old, known for at least twenty years, and one may wonder why they have not before been put to use in connection with the Riemann moduli space. Maybe we lacked the inspiration that comes from the renewed interaction with physics, exemplified in conformal field theories.

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1. Spaces of surfaces

1.1. Moduli and mapping classes. Fix a closed smooth and oriented surface F of genus g. One way to define the moduli space $\mathcal{M}_g$ is to start with the set of almost complex structures on F, compatible with the orientation. Such a structure is a fibrewise map $J\colon TF \to TF$ of the tangent bundle with $J^2 = -\mathrm{id}$ and with the property that $\{v, Jv\}$ is an oriented basis for each non-zero tangent vector v. The map J can be thought of as a section of the fibre bundle associated to TF with fibre $\mathrm{GL}_2^+(\mathbb{R})/\mathrm{GL}_1(\mathbb{C})$. We topologize this section space by the C^∞ Whitney topology and denote it $\mathcal{S}_{\mathbb{C}}(TF)$. The group $\mathrm{Diff}(F)$ of orientation preserving diffeomorphisms of F acts on $\mathcal{S}_{\mathbb{C}}(TF)$. The orbit space is the moduli space

$$\mathcal{M}_g = \mathcal{M}(F) = \mathcal{S}_{\mathbb{C}}(TF)/\mathrm{Diff}(F).$$

By a theorem of Gauss, $\mathcal{S}_{\mathbb{C}}(TF)$ is equal to the set of maximal holomorphic atlasses on F that respect the orientation: elements of $\mathcal{S}_{\mathbb{C}}(TF)$ are Riemann surfaces with underlying manifold F.

The moduli space $\mathcal{M}_0$ is a single point by Riemann's mapping theorem, $\mathcal{M}_1 = \mathbb{R}^2$, so we can concentrate on $\mathcal{M}_g$ for $g \geq 2$, where the moduli space is not contractible. Any Riemann surface Σ of genus $g \geq 2$ is covered by the upper half plane $H \subset \mathbb{C}$, so it is a holomorphic space form $\Sigma = H/\Gamma$ with Γ a cocompact torsion free subgroup of the group $\mathrm{PSL}_2(\mathbb{R})$ of all Möbius transformations of H. $\mathrm{PSL}_2(\mathbb{R})$ is also the group of all isometries of H in its standard hyperbolic metric $ds^2 = |dz|^2/y^2$, so Σ is a hyperbolic space form as well, and $\mathcal{S}_{\mathbb{C}}(TF)$ could be replaced with the space of hyperbolic metrics in the definition of $\mathcal{M}_g$, $g \geq 2$.

The connected component $\mathrm{Diff}_1(F)$ of the identity acts freely on $\mathcal{S}_{\mathbb{C}}(TF)$ by an easy fact from hyperbolic geometry. The associated orbit space

$$\mathcal{T}_g = \mathcal{T}(F) = \mathcal{S}_{\mathbb{C}}(TF)/\mathrm{Diff}_1(F)$$

is the Teichmüller space. It is homeomorphic to $\mathbb{R}^{6g-6}$. The rest of the $\mathrm{Diff}(F)$ action on $\mathcal{S}_{\mathbb{C}}(TF)$ is the action of the mapping class group,

$$\Gamma_g = \Gamma(F) = \pi_0 \mathrm{Diff}(F),$$

on $\mathcal{T}_g$. It acts discontinuously with finite isotropy groups, so $\mathcal{M}_g = \mathcal{T}_g/\Gamma_g$ is a $(6g-6)$-dimensional orbifold. Had the action been free, then $\mathcal{M}_g$ would have been homotopy equivalent to the space $B\Gamma_g$, classifying Γ_g-covering spaces. As it is, there is a map from $B\Gamma_g$ to $\mathcal{M}_g$ that induces isomorphisms

$$H^*(\mathcal{M}_g;\mathbb{Q}) \xrightarrow{\cong} H^*(B\Gamma_g;\mathbb{Q}) \tag{1.1}$$

on rational cohomology.

Because $\mathcal{S}_{\mathbb{C}}(TF)$ is the space of sections in a fibre bundle with contractible fibre $\mathrm{GL}_2^+(\mathbb{R})/\mathrm{GL}_1(\mathbb{C})$, it is itself contractible. Earle and Eells [15] proved that the orbit map of the $\mathrm{Diff}_1(F)$ action,

$$\pi : \mathcal{S}_{\mathbb{C}}(TF) \to \mathcal{T}(F)$$

locally has a section, so that π is a fibre bundle. The total space and the base space both being contractible, they concluded that $\mathrm{Diff}_1(F)$ (and hence any other connected component of $\mathrm{Diff}(F)$) is contractible. This gives the homotopy equivalence

$$B\,\mathrm{Diff}(F) \xrightarrow{\simeq} B\Gamma(F) \tag{1.2}$$

of classifying spaces; cf. §2.1 below for a general discussion of classifying spaces.

The model of $B\,\mathrm{Diff}(F)$ that we use is the space of all oriented surfaces of euclidean space (of arbitrary dimension) that are diffeomorphic to F. This is equal to the orbit space

$$\mathrm{Emb}(F, \mathbb{R}^\infty)/\,\mathrm{Diff}(F)$$

of the free $\mathrm{Diff}(F)$ action on the space of smooth embeddings of F in infinite dimensional euclidean space. The embedding space is contractible [80] and the orbit map is a fibre bundle. This implies the homotopy equivalence

$$\mathrm{Emb}(F, \mathbb{R}^\infty)/\,\mathrm{Diff}(F) \simeq B\,\mathrm{Diff}(F).$$

Each surface in euclidean space inherits a Riemannian metric from the surroundings, which together with the orientation defines, a complex structure. This leads to a concrete map

$$\mathrm{Emb}(F, \mathbb{R}^\infty)/\,\mathrm{Diff}(F) \to \mathcal{M}(F)$$

that induces isomorphism on rational cohomology.

1.2. Stability and Mumford's standard conjecture. Ideally, one would like to compute the rational cohomology ring of each individual $\mathcal{M}_g$. This has been done for $g \leq 4$ in [44], [72], but seems too ambitious for larger genus. Following Mumford [57], one should instead attempt a partial calculation of $H^*(\mathcal{M}_g)$, namely in a certain stable range. For $g \geq 2$, Mumford defined tautological classes in the rational Chow ring of the Deligne–Mumford compactification $\overline{\mathcal{M}}_g$, and proposed that their images in $H^*(\mathcal{M}_g; \mathbb{Q})$ freely generate the entire rational cohomology ring as $g \to \infty$. The proposal is similar in spirit to what happens for the Grassmannian of d-dimensional linear subspaces of $\mathbb{C}^n$; as $n \to \infty$, the cohomology becomes a polynomial algebra in the Chern classes of the tautological d-dimensional vector bundle. See also [28], [41].

More precisely, Mumford predicted that, in a range of dimensions that tends to infinity with g, the cohomology ring $H^*(\mathcal{M}_g; \mathbb{Q})$ is isomorphic with the polynomial algebra $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$ in the tautological classes κ_i of degree $2i$.

Miller [52] and Morita [56] used topological methods to define integral cohomology classes in $B\Gamma_g$ that agree with Mumford's classes under the isomorphism (1.1). I recall the definition. Choose a point $p \in F$ and consider the subgroup $\mathrm{Diff}(F;\, p)$ of orientation-preserving diffeomorphisms that fixes p. It has contractible components [16] and mapping class group $\Gamma_g^1 = \pi_0 \mathrm{Diff}(F;\, p)$. In the diagram

$$\begin{array}{ccccc} B\,\mathrm{Diff}(F;\, p) & \longrightarrow & B\Gamma_g^1 & \longrightarrow & \mathcal{M}_g^1 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ B\,\mathrm{Diff}(F) & \longrightarrow & B\Gamma_g & \longrightarrow & \mathcal{M}_g, \end{array}$$

the left-hand horizontal maps are homotopy equivalences and the right-hand ones are rational cohomology isomorphisms. The right-hand vertical map is the "universal curve"; the left-hand π is (homotopic to) a smooth fibre bundle with fibre F and oriented relative tangent bundle $T\pi$. Morita defines

$$\kappa_i = (-1)^{i+1}\pi_!(c_1(T\pi)^{i+1}) \in H^{2i}(B\Gamma_g;\, \mathbb{Z}), \tag{1.3}$$

where $\pi_!$ is the Gysin (or integration along the fibres) homomorphism.

The "differential at p" gives a map from $\mathrm{Diff}(F;\, p)$ to $\mathrm{GL}_2^+(\mathbb{R})$, and the associated map

$$d\colon B\,\mathrm{Diff}(F;\, p) \to B\,\mathrm{GL}_2^+(\mathbb{R}) \simeq \mathbb{C}P^\infty$$

classifies $T\pi$. Its (homotopy theoretic) fibre is the classifying space of the group $\mathrm{Diff}(F;\, D(p))$ of orientation-preserving diffeomorphisms that fix the points of a small disc $D(p)$ around p. Let $\Gamma_{g,1} = \pi_0 \mathrm{Diff}(F;\, D(p))$ be its mapping class group so that we have the fibration

$$B\Gamma_{g,1} \xrightarrow{\ \pi\ } B\Gamma_g^1 \xrightarrow{\ d\ } \mathbb{C}P^\infty$$

to relate the cohomology of $B\Gamma_{g,1}$ and $B\Gamma_g^1$. Note that $\mathrm{Diff}(F,\, D(g)) = \mathrm{Diff}(F_{g,1};\, \partial)$, where $F_{g,1} = F - \mathrm{int}\, D(p)$ is a genus g surface with one boundary circle. Since $F_{g+1,1}$ is the union of $F_{g,1}$ with a torus $F_{1,2}$ with two boundary circles, there is a map $\Gamma_{g,1} \to \Gamma_{g+1,1}$. Forgetting $D(p) \subset F$ (or filling out the hole in $F_{g,1}$) gives a map $\Gamma_{g,1} \to \Gamma_g$. The following theorem [27] with an improvement from [38] is of crucial importance to us. This is Harer's stability theorem.

Theorem 1.1 ([27], [38]). *The induced maps*

$$H_k(B\Gamma_{g,1};\, \mathbb{Z}) \to H_k(B\Gamma_{g+1,1};\, \mathbb{Z}) \quad \textit{and} \quad H_k(B\Gamma_{g,1};\, \mathbb{Z}) \to H_k(B\Gamma_g;\, \mathbb{Z})$$

are isomorphisms when $2k < g - 1$.

The stable mapping class group $\Gamma_{\infty,1}$ is the direct limit of the groups $\Gamma_{g,1}$ as $g \to \infty$. By the theorem,

$$H^k(B\Gamma_{\infty,1};\mathbb{Z}) \cong H^k(B\Gamma_g;\mathbb{Z})$$

when $2k < g-1$. Miller and Morita proved in [52] and [56] that $H^*(B\Gamma_{\infty,1};\mathbb{Q})$ contains the polynomial algebra $\mathbb{Q}[\kappa_1,\kappa_2,\ldots]$.

Let $\mathbb{C}P^n$ denote the complex projective n-space, and let $L_n^\perp \subset \mathbb{C}P^n \times \mathbb{C}^{n+1}$ be the subspace of pairs (l, v) with v orthogonal to l. Consider the space $MT(n)$ of all proper maps from $\mathbb{C}^{n+1}$ to $L_n^\perp$. The cohomology groups $H^k(MT(n);\mathbb{Z})$ are independent of n for $k < 2n$. One form of the generalized Mumford conjecture is the statement

Theorem 1.2 ([47]). *For $k < 2n$ there is an isomorphism*

$$H^k(B\Gamma_{\infty,1};\mathbb{Z}) \cong H^k(MT(n);\mathbb{Z}).$$

Corollary 1.3 ([47]). *The rational cohomology ring of $B\Gamma_{\infty,1}$ is a polynomial algebra in the classes $\kappa_1,\kappa_2,\ldots$.*

In view of (1.1), this also calculates $H^*(\mathcal{M}_g;\mathbb{Q})$ for a range of dimensions, and affirms Mumford's conjecture.

The stability theorem from [27], [38] is more general than stated above. Let $F^s_{g,b}$ be a surface of genus g with $b \geq 0$ boundary circles and s distinct points in the interior, and let $\Gamma^s_{g,b}$ be the associated mapping class group.

Addendum 1.4. For $b > 0$, the maps

$$B\Gamma^s_{g,b-1} \leftarrow B\Gamma^s_{g,b} \rightarrow B\Gamma^s_{g+1,b}$$

induce isomorphisms in integral cohomology in degrees less than $(g-1)/2$.

The addendum implies that $H^*(B\Gamma^s_{\infty,b};\mathbb{Z})$ is independent of the number of boundary circles. Consequently, we sometimes drop the subscript b from the notation and write Γ^s_∞ instead of $\Gamma^s_{\infty,b}$. In the diagram

$$\begin{tikzcd}
B\Gamma_{g,b+s} \arrow[r] \arrow[dr] & B\Gamma^s_{g,b} \arrow[r, "d"] \arrow[d] & \prod^s \mathbb{C}P^\infty \\
& B\Gamma_{g,b}\,, &
\end{tikzcd}$$

the skew map is a cohomology isomorphism in the stability range, so

$$H^*(B\Gamma^s_{\infty,b};\mathbb{Z}) = H^*(B\Gamma_{\infty,1};\mathbb{Z}) \otimes \mathbb{Z}[\psi_1,\ldots,\psi_s] \tag{1.4}$$

with $\deg \psi_i = 2$. See also [6].

2. Cobordism categories and their spaces

In this section we explain work of S. Galatius, U. Tillmann, M. Weiss and the author in various combinations. The most relevant references are [22], [23], [46], [47]. The section contains, in outline, a proof of the generalized Mumford conjecture, different from the original one, but still based on concepts and results from [47].

2.1. The classifying space of a category. In [53], Milnor associated to each topological group G a space BG by a functorial construction, characterized up to homotopy by being the base of a principal G-bundle with contractible total space. Moreover, isomorphim classes of principal G-bundles with base X are in one to one correspondence with homotopy classes of maps from X to BG.

The space $B\mathcal{C}$ associated to a (small) category $\mathcal{C}$ is a similar construction [64]. The objects of $\mathcal{C}$ are the vertices in a simplicial set. Two objects span a 1-simplex if there is a morphism between them. A k-simplex corresponds to k composable arrows of $\mathcal{C}$. Formally, let $N_k\mathcal{C}$ be the set of k-tuples of morphisms,

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \longrightarrow \cdots \xrightarrow{f_k} c_k,$$

and define face operators

$$d_i\colon N_k\mathcal{C} \to N_{k-1}\mathcal{C}, \quad i = 0, 1, \dots, k$$

by removing c_i (and composing f_i and f_{i+1} when $i \neq 0, k$). Then

$$B\mathcal{C} = \coprod_{k=0}^{\infty} \Delta^k \times N_k\mathcal{C} \big/ (d^i t, f) \equiv (t, d_i f) \tag{2.1}$$

with $t \in \Delta^{k-1}$ and $f \in N_k\mathcal{C}$. Here Δ^k is the standard euclidean k-simplex and $d^i\colon \Delta^{k-1} \to \Delta^k$ the inclusion as the i'th face.

The categories we use below are *topological* categories. This means that the total set of objects and the total set of morphisms have topologies, and that the structure maps (source, target and composition) are continuous. In this case $N_k\mathcal{C}$ is a space and d_i is continuous. I refer to [79] for a discussion of the kind of objects that $B\mathcal{C}$ actually classifies.

A topological category M with a single object is precisely a topological monoid. In this case, the 1-skeleton in (2.1) gives a map from $\Delta^1 \times M$ into BM, or equivalently a map from M into the loop space ΩBM. It takes the monoid $\pi_0 M$ of connected components into the fundamental group $\pi_1 BM$. If $\pi_0 M$ is a group then $M \to \Omega BM$ turns out to be a homotopy equivalence. More generally, we have the group completion theorem that goes back to Quillen's work in K-theory. The composition law in M yields a ring structure on $H_*(M)$, and we have

Theorem 2.1 ([4], [51]). *Suppose that $\pi_0 M$ is central in $H_*(M)$. Then*

$$H_*(M)[\pi_0 M^{-1}] \xrightarrow{\cong} H_*(\Omega BM)$$

is an isomorphism.

In typical applications of Theorem 2.1, $\pi_0 M = \mathbb{N}$ and the left-hand side has the following interpretation. Let $m \in M$ represent $1 \in \pi_0 M$ and define M_∞ to be the direct limit of $M \xrightarrow{m} M \xrightarrow{m} M \xrightarrow{m} \cdots$. Then $H_*(M)[\pi_0 M^{-1}]$ is the homology of $\mathbb{Z} \times M_\infty$.

Remark 2.2. In general, the direct limit (or colimit) of a string of spaces $f_i \colon X_i \to X_{i+1}$ does not commute with homology, unless the maps f_i are closed injections. When this is not the case, the colimit should be replaced with the homotopy colimit (or telescope) of [54]. This construction always commutes with homology. The right definition of M_∞ is therefore

$$M_\infty = \operatorname{hocolim}(M \xrightarrow{m} M \xrightarrow{m} M \xrightarrow{m} \cdots).$$

Alternatively, we can apply Quillen's plus-construction [5] to M_∞ to get a homotopy equivalence

$$\mathbb{Z} \times M_\infty^+ \xrightarrow{\simeq} \Omega BM. \tag{2.2}$$

This applies to the monoids $M = \coprod B\Sigma_n$, $\coprod B\,\mathrm{GL}_n(R)$ or $\coprod B\Gamma_{g,2}$ where the composition law is induced from the direct sum of permutations and matrices and, in the case of the mapping class group, from gluing along one boundary circle. There are homotopy equivalences

$$\begin{aligned} \mathbb{Z} \times B\Sigma_\infty^+ &\xrightarrow{\simeq} \Omega B\big(\textstyle\coprod B\Sigma_n\big), \\ \mathbb{Z} \times B\,\mathrm{GL}_\infty(R)^+ &\xrightarrow{\simeq} \Omega B\big(\textstyle\coprod B\,\mathrm{GL}_n(R)\big), \\ \mathbb{Z} \times B\Gamma_{\infty,2}^+ &\xrightarrow{\simeq} \Omega B\big(\textstyle\coprod B\Gamma_{n,2}\big). \end{aligned} \tag{2.3}$$

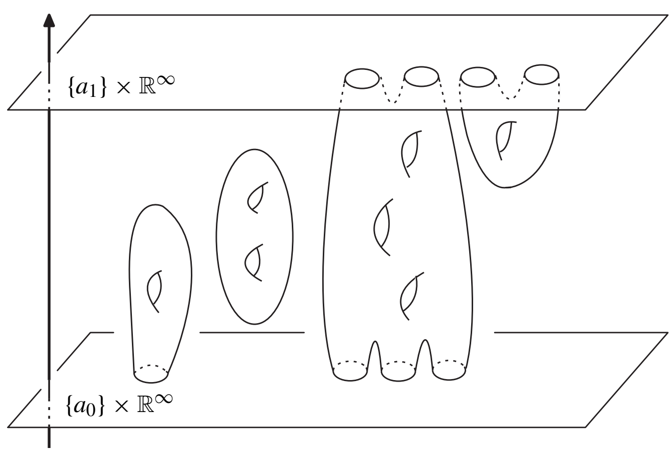
The effect of the plus-construction is quite dramatic. While it leaves homology groups unchanged, it produces extra homotopy groups. The spaces $B\Sigma_\infty$, $B\,\mathrm{GL}_\infty(R)$, and $B\Gamma_{\infty,2}$ have vanishing homotopy groups past the fundamental group, but their plus-constructions have very interesting homotopy groups: $\pi_i B\Sigma_\infty^+$ is the i'th stable homotopy group of spheres, and $\pi_i B\,\mathrm{GL}_\infty(R)^+$ is Quillen's higher K-group $K_i(R)$ [61], [62].

2.2. Riemann's surface category and generalizations. The Riemann surface category $\mathcal{S}$ has attracted attention with G. Segal's treatment of conformal field theories [65], [66]. It is the category with one object C_m for each non-negative integer m, namely the disjoint union of m parametrized circles. A morphism from C_m to C_n

is a pair (Σ, φ), consisting of a Riemann surface Σ and an orientation-preserving diffeomorphism $\varphi\colon \partial\Sigma \to (-C_m) \sqcup C_n$. (The topology on the set of morphisms is induced from the topology of the moduli spaces.)

As indicated at the end of §1.1, one may replace the moduli space $\mathcal{M}_g$ with the space of surfaces in euclidean space without changing the rational homology. For $\mathcal{M}_{g,b}$ with $b > 0$, the replacement does not even change the homotopy type. This leads to the category $\mathcal{C}_2$ of embedded surfaces. Once we go to embedded surfaces instead of Riemann surfaces, there are no added complications in generalizing from 2 dimensions to d dimensions. This leads to the category $\mathcal{C}_d$.

An object of $\mathcal{C}_d$ is a $(d-1)$-dimensional closed, oriented submanifold of $\{a\} \times \mathbb{R}^{n+d-1} \subset \mathbb{R}^{n+d}$ for some real number a and some large n. A morphism is a compact, oriented d-dimensional manifold W^d contained in a strip $[a_0, a_1] \times \mathbb{R}^{n+d-1}$ $(a_0 < a_1)$ such that $\partial W = (-\partial_0 W) \sqcup \partial_1 W$, where $\partial_i W = W \cap \{a_i\} \times \mathbb{R}^{n+d-1}$. For technical reasons, we require that W meets the walls $\{a_i\} \times \mathbb{R}^{n+d-1}$ orthogonally and that W is constant near the walls. Here is a schematic picture of W:



The number n is arbitrarily large, and not part of the structure. From now on, I often write $n = \infty$. A submanifold of $\mathbb{R}^{\infty+d-1}$ can be parametrized in the sense that it is the image of an embedding. Thus we have the identifications

$$N_0\mathcal{C}_d \cong \coprod_{\{M\},a} \operatorname{Emb}(M^{d-1}, \{a\} \times \mathbb{R}^{\infty+d-1})/\operatorname{Diff}(M), \tag{2.4a}$$

$$N_1\mathcal{C}_d \cong \Big(\coprod_{\{W\},a_0<a_1} \operatorname{Emb}(W^d, [a_0, a_1] \times \mathbb{R}^{\infty+d-1})/\operatorname{Diff}(W^d)\Big) \sqcup N_0\mathcal{C}_d. \tag{2.4b}$$

The disjoint unions vary over the set of diffeomorphism classes of smooth compact, oriented manifolds and over real numbers a, a_i. The embedding spaces are equipped with the "convenient topology", [42]; the action of the diffeomorphism groups is by composition. The quotient maps are principal fibre bundles [42], and the embedding spaces are contractible, so the individual terms in (2.4) are homotopy equivalent to $B\,\mathrm{Diff}(M^{d-1})$ and $B\,\mathrm{Diff}(W^d) \sqcup B\,\mathrm{Diff}(M^{d-1})$, respectively. See also [46, §2].

The group of connected components $\pi_0 B\mathcal{C}_d$ is equal to the cobordism group Ω_{d-1} of oriented closed $(d-1)$-manifolds. This group has been tabulated for all d [78]. It vanishes for $d \leq 4$.

Proposition 2.3 ([22]). *The classifying space $B\mathcal{C}_2$ has the same rational homology as the classifying space $B\mathcal{S}$ of the Riemann surface category.*

2.3. Thom spaces and embedded cobordisms. The Thom space $\mathrm{Th}(\xi)$ (sometimes denoted X^ξ) of a vector bundle ξ on X is a construction that has been of fundamental importance in topology for more than fifty years. The homotopy theory of specific Thom spaces has helped solve many geometric problems. Early results can be found in [59], [60], [69], [70]. Our use of Thom spaces are not far from this original tradition. The modern development is described in [35].

For a vector bundle ξ over a compact base space, $\mathrm{Th}(\xi)$ is the one point compactification of its total space. Equivalently, it is the quotient of the projective bundle $P(\xi \oplus \mathbb{R})$ by $P(\xi)$.

Two geometric properties might help explain the usefulness of the construction. First, the complement $\mathrm{Th}(\xi) - X$ of the zero section is contractible so that $\mathrm{Th}(\xi)$ is a kind of homotopy theoretic localization of ξ near X. Second, if ξ is the normal bundle of a submanifold $X^m \subset \mathbb{R}^{m+k}$, then one has the Pontryagin–Thom collapse map,

$$c_X \colon S^{m+k} \longrightarrow \mathrm{Th}(\xi),$$

by considering ξ to be an open tubular neighborhood of X in $\mathbb{R}^{m+k}$.

On the algebraic side, we have the Thom isomorphism

$$H^i(X;\mathbb{Z}) \cong H^{i+k}(\mathrm{Th}(\xi);\mathbb{Z})$$

provided that ξ is an oriented vector bundle of dimension k.

Let $G(d,n)$ denote the Grassmannian of *oriented* d-dimensional linear subspaces of $\mathbb{R}^{d+n}$, and let

$$\begin{aligned} U_{d,n} &= \{(V,v) \in G(d,n) \times \mathbb{R}^{d+n} \mid v \in V\},\\ U^\perp_{d,n} &= \{(V,v) \in G(d,n) \times \mathbb{R}^{d+n} \mid v \perp V\} \end{aligned}$$

be the two canonical vector bundles over it.

The restriction of $U^\perp_{d,n+1}$ to $G(d,n)$ is the product $\mathbb{R} \times U^\perp_{d,n}$. Its inclusion into $U^\perp_{d,n+1}$ is a proper map, so it induces a map $\varepsilon_{d,n}$ of one point compactifications from the suspension $\Sigma\,\mathrm{Th}(U^\perp_{d,n})$ to $\mathrm{Th}(U^\perp_{d,n+1})$.

At this point, it is convenient to introduce the concept of a prespectrum $E = \{E_n, \varepsilon_n\}$. It consists of a sequence of pointed spaces E_n for $n = 0, 1, \ldots$ together with maps $\varepsilon_n\colon \Sigma E_n \to E_{n+1}$, where

$$\Sigma E_n = S^1 \times E_n/(* \times E_n \cup S^1 \times *).$$

The infinite loop space associated with E is defined to be

$$\Omega^\infty E = \operatorname{hocolim}(\cdots \to \Omega^n E_n \xrightarrow{\varepsilon_n'} \Omega^{n+1} E_{n+1} \to \cdots)$$

with ε_n' being the adjoint of ε_n. We shall also need its deloop $\Omega^{\infty-1}E$. This is the homotopy colimit of $\Omega^{n-1}E_n$.

The pairs $\{\mathrm{Th}(U_{d,n}^\perp), \varepsilon_{d,n}\}$ form the prespectrum $\mathbb{G}_{-d}$; its n'th space is $\mathrm{Th}(U_{d,n-d}^\perp)$ if $n \geq d$ and otherwise a single point. The sphere prespectrum $\mathbb{S}$ has the n-sphere as its n'th space and ε_n is the canonical identification $\Sigma S^n = S^{n+1}$. The associated infinite loop spaces are

$$\Omega^\infty \mathbb{S} = \operatorname{hocolim} \Omega^n S^n, \quad \Omega^\infty \mathbb{G}_{-d} = \operatorname{hocolim} \Omega^{n+d}\,\mathrm{Th}(U_{d,n}^\perp).$$

In both cases, the inclusion of the n'th term into the limit space induces an isomorphism on homology groups in a range of dimensions that tends to infinity with n. The space $G(2,\infty)$ is homotopy equivalent to $\mathbb{C}P^\infty$, and $\mathbb{G}_{-2}$ becomes homotopy equivalent to the prespectrum $\mathbb{C}P_{-1}^\infty$. This has $(2n+2)$nd space equal to the Thom space $\mathrm{Th}(L_n^\perp)$ of the complement $L_n^\perp$ to the canonical line bundle over $\mathbb{C}P^n$. The associated infinite loop spaces are homotopy equivalent,

$$\Omega^\infty \mathbb{G}_{-2} \simeq \Omega^\infty \mathbb{C}P_{-1}^\infty. \tag{2.5}$$

Theorem 2.4 ([22]). *For $d \geq 0$, $B\mathcal{C}_d$ is homotopy equivalent to $\Omega^{\infty-1}\mathbb{G}_{-d}$.*

A few words of explanation are in order. Suppose that $W^d \subset [a_0, a_1] \times \mathbb{R}^{n+d-1}$ is a morphism of $\mathcal{C}_d$. For each $p \in W$, the tangent space T_pW is an element of $G(d,n)$ and the normal space at p is precisely the fibre of $U_{d,n}^\perp$ at T_pW. The Pontryagin–Thom collapse map

$$[a_0, a_1] \times (S^{n+d-1}, \infty) \to (\mathrm{Th}(U_{d,n}^\perp), \infty)$$

defines a path in $\Omega^{n+d-1}\,\mathrm{Th}(U_{d,n}^\perp)$, and hence for $n \to \infty$, a path in $\Omega^{\infty-1}\mathbb{G}_{-d}$. More generally, an element of $N_k\mathcal{C}_d$ gives a set of k composable paths in $\Omega^{\infty-1}\mathbb{G}_{-d}$.

To any space X, one can associate the path category Path X. Its space of objects is $\mathbb{R} \times X$, and a morphism from (a_0, x_0) to (a_1, x_1) is a path $\gamma\colon [a_0, a_1] \to X$ from x_0 to x_1. The classifying space of Path X is homotopy equivalent to X, $B\,\mathrm{Path}\,X \simeq X$. In the situation above, this leads to a well-defined homotopy class

$$\beta_d\colon B\mathcal{C}_d \longrightarrow B(\mathrm{Path}\,\Omega^{\infty-1}\mathbb{G}_{-d}) \simeq \Omega^{\infty-1}\mathbb{G}_{-d},$$

and Theorem 2.4 is the statement that β_d is a homotopy equivalence.

I shall attempt to explain the strategy of proof of Theorem 2.4 by breaking it down into its three major parts.

Let $\mathcal{T}_d(\mathbb{R}^{n+d})$ be the space of oriented d-dimensional submanifolds $E^d \subset \mathbb{R}^{n+d}$ contained in a tube $\mathbb{R} \times D^{n+d-1}$ as a closed subset. Let $\mathcal{T}_d(\mathbb{R}^{\infty+d})$ be the union of these spaces.

We topologize $\mathcal{T}_d(\mathbb{R}^{n+d})$, not by the Whitney embedding topology used above, but by a coarser topology that allows manifolds to be pushed to infinity. The topology we want has the property that a map from a k-dimensional manifold X^k into $\mathcal{T}_d(\mathbb{R}^{n+d})$ produces a submanifold M^{k+d} of $X^k \times \mathbb{R} \times \mathbb{R}^{n+d-1}$, such that the projection onto $X \times \mathbb{R}$ is proper and the projection on X is a submersion (and not necessarily a fibre bundle).

There is a partially ordered set $\mathcal{D}_d$ associated with $\mathcal{T}_d(\mathbb{R}^{\infty+d})$. It consists of pairs (a, E) with $a \in \mathbb{R}$, $E \in \mathcal{T}_d(\mathbb{R}^{\infty+d})$, such that E intersects the wall $a \times \mathbb{R}^{\infty+d-1}$ orthogonally. The partial ordering is that $(a_0, E_0) \leq (a_1, E_1)$ if $a_0 \leq a_1$ and $E_0 = E_1$. A partially ordered set is a category with one arrow for each order relation $(a_0, E_0) \leq (a_1, E_1)$.

Three maps connect the spaces involved:

$$r\colon \mathcal{D}_d \to \mathcal{C}_d, \quad s\colon \mathcal{D}_d \to \mathcal{T}_d(\mathbb{R}^{\infty+d}), \quad t\colon \mathcal{T}_d(\mathbb{R}^{\infty+d}) \to \Omega^{\infty-1}\mathbb{G}_{-d}.$$

The map r intersects E^d with the wall $\{a\} \times \mathbb{R}^{\infty+d-1}$, and the morphism $(a_0, E) \leq (a_1, E)$ with the strip $[a_0, a_1] \times \mathbb{R}^{\infty+d-1}$; s forgets the first coordinate, and t is the Pontryagin–Thom collapse map. The proof is now to show that each of the induced maps

$$r\colon B\mathcal{D}_d \to B\mathcal{C}_d, \tag{2.6a}$$

$$s\colon B\mathcal{D}_d \to \mathcal{T}_d(\mathbb{R}^{\infty+d}), \tag{2.6b}$$

$$t\colon \mathcal{T}_d(\mathbb{R}^{\infty+d}) \to \Omega^{\infty-1}\mathbb{G}_{-d} \tag{2.6c}$$

are a homotopy equivalences. It suffices to check on homotopy groups, i.e. that $\pi_n(r)$, $\pi_n(s)$, and $\pi_n(t)$ are isomorphisms. This is done geometrically as in [47], by interpreting the homotopy groups of the given spaces as cobordism classes of families of the involved structures indexed by the sphere S^n.

It is (2.6c) that requires an h-principle from singularity theory. Given an element of $\pi_n(\Omega^{\infty-1}\mathbb{G}_{-d})$ one uses transversality together with Phillips' submersion theorem [58] to obtain a cobordism class of triples (E^{n+d}, π, f), where $\pi\colon E^{n+d} \to S^n$ is a submersion, and $f\colon E^{n+d} \to \mathbb{R}$ is a proper map. This represents an element of $\pi_n\mathcal{T}_d(\mathbb{R}^{\infty+d})$. Injectivity is proved by relative considerations.

Remark 2.5. The d-fold suspensions $\Sigma^d\mathbb{G}_{-d}$ fit together via maps $\Sigma^d\mathbb{G}_{-d} \to \Sigma^{d+1}\mathbb{G}_{-(d+1)}$. Their homotopy colimit is the prespectrum MSO whose homotopy groups are the cobordism groups Ω_*, by [70].

2.4. Consequences of Harer stability. For clarity, I begin with a discussion of abstract stability in a topological category $\mathcal{C}$, generalizing the group completion theorem.

Given a string of morphisms in $\mathcal{C}$,

$$b_1 \xrightarrow{\beta_1} b_2 \xrightarrow{\beta_2} b_3 \longrightarrow \cdots,$$

we have functors

$$F_i : \mathcal{C}^{\mathrm{op}} \to \text{spaces}; \quad F_\beta : C^{\mathrm{op}} \to \text{spaces}$$

with $F_i(c) = \mathcal{C}(c, b_i)$, the space of morphisms from c to b_i, and

$$F_\beta(c) = \operatorname{hocolim}(F_1(c) \xrightarrow{\beta_1} F_2(c) \xrightarrow{\beta_2} F_3(c) \longrightarrow \cdots).$$

Lemma 2.6. *Suppose $B\mathcal{C}$ is connected, and suppose for each morphism $c_1 \to c_2$ in $\mathcal{C}$ that $F_\beta(c_2) \to F_\beta(c_1)$ is an integral homology isomorphism. Then there is an integral homology isomorphism $F_\beta(c) \to \Omega B\mathcal{C}$ for each object c of $\mathcal{C}$.*

This follows from [51]: Consider the category $F_\beta \wr \mathcal{C}$. Its objects are pairs (x, c) with $x \in F_\beta(c)$, and it has the obvious morphisms. Its classifying space is contractible because it is the homotopy colimit of the contractible spaces $B(F_i \wr \mathcal{C})$†. The map $\pi : B(F_\beta \wr \mathcal{C}) \to B\mathcal{C}$ is a homology fibration. The fibre $\pi^{-1}(c) = F_\beta(c)$ is therefore homology equivalent to the homotopy fibre, which is $\Omega B\mathcal{C}$.

The condition of Lemma 2.6 is not satisfied for the embedded surface category $\mathcal{C}_2$ of §2.3, but it is satisfied for a certain subcategory $\mathcal{C}_2^{\mathrm{res}} \subset \mathcal{C}_2$, originally introduced in [71] for that very reason.

The restricted category $\mathcal{C}_d^{\mathrm{res}} \subset \mathcal{C}_d$ has the same space of objects but a restricted space of morphisms: A morphism $W^d \subset [a_0, a_1] \times \mathbb{R}^{\infty+d-1}$ of $\mathcal{C}_d$ belongs to $\mathcal{C}_d^{\mathrm{res}}$ if each connected component of W has a non-empty intersection with $\{a_1\} \times \mathbb{R}^{\infty+d-1}$.

Theorem 2.7 ([22]). *For $d > 1$, $B\mathcal{C}_d^{\mathrm{res}} \to B\mathcal{C}_d$ is a homotopy equivalence.*‡

The proof is, roughly speaking, to perform surgery (connected sum) on morphisms of $\mathcal{C}_d$ to replace them with morphisms from $\mathcal{C}_d^{\mathrm{res}}$.

Given Theorem 2.4 and Theorem 2.7, and using the notation (2.5), we can adopt Tillmann's argument [71] to prove the generalized Mumford conjecture,

Theorem 2.8 ([47]). *The space $\mathbb{Z} \times B\Gamma_{\infty,1}^{+}$ is homotopy equivalent to the space $\Omega^\infty \mathbb{C}P_{-1}^\infty$.*

We take $\mathcal{C} = \mathcal{C}_2^{\mathrm{res}}$ and b_i to be the object consisting of a standard circle in $\{i\} \times \mathbb{R}^{\infty+1}$. The morphism β_i is the torus $F_{1,2}$ with two boundary circles embedded in $[i, i+1] \times \mathbb{R}^{\infty+1}$ so that $\partial\beta_i = b_{i+1} \sqcup -b_i$. Then

$$F_\beta(c) \simeq \mathbb{Z} \times B\Gamma_{\infty,|c|+1},$$

† $F_i \wr \mathcal{C}$ has the terminal object (b_i, id).

‡The theorem is almost certainly valid also for $d = 1$, but the present proof works only for $d > 1$.

where $|c|$ is the number of components in the object c. Addendum 1.4 and Lemma 2.6 apply.

Remark 2.9. The analogue of Theorem 2.8 has been established for the spin mapping class group, and for the non-orientable mapping class group in [21] and [73], respectively. The stable cohomology in the spin case differs only from the orientable case in 2-torsion. In the non-orientable case, the stable rational cohomology is a polynomial algebra in $4i$-dimensional classes.

2.5. Cohomology of $\Omega^\infty\mathbb{G}_{-d}$. The cohomology groups of a prespectrum E are defined to be inverse limits of the cohomology of the individual terms

$$H^k(E) = \varprojlim_n H^{k+n}(E_n; *), \tag{2.7}$$

with the maps in the inverse limit induced by ε_n. Note that $H^k(E)$ might be non-zero also for negative values of k. For example we have

$$\begin{aligned} H^k(\mathbb{S}) &= \mathbb{Z} \qquad \text{for } k = 0, \\ H^k(\mathbb{S}) &= 0 \qquad \text{for } k \neq 0, \end{aligned}$$

and

$$H^k(\mathbb{G}_{-d}) = H^{k+d}(G(d,\infty)).$$

The homotopy groups and homology groups of E are the direct limits

$$\pi_k E = \operatorname{colim} \pi_{k+n}(E_n; *), \quad H_k E = \operatorname{colim} H_{k+n}(E_n; *).$$

Given a spectrum $E = \{E_n, \varepsilon_n\}$ and a space X, we can form the spectrum

$$E \wedge X_+ = \{E_n \wedge X_+, \varepsilon \wedge \mathrm{id}_X\} \quad (E_n \wedge X_+ = E_n \times X/* \times X)$$

and its associated infinite loop space $\Omega^\infty(E \wedge X_+)$. The homotopy groups

$$E_*(X) = \pi_*(E \wedge X_+)$$

form a generalized homology theory: they satisfy the axioms of usual homology, save the dimensional axiom that $H_k(\mathrm{pt})$ vanishes for $k \neq 0$. We shall apply the construction in §3.2 with $E = \mathbb{C}P^\infty_{-1}$.

The cohomology groups of $\Omega^\infty E$ and the cohomology groups of E, as defined in (2.7), are related by the cohomology suspension homomorphism

$$\sigma^*\colon H^k(E) \to H^k(\Omega^\infty_0 E),$$

where $\Omega^\infty_0 E$ denotes the component of the trivial loop†. The suspension σ^* is induced from the evaluation map from $\Sigma^n\Omega^n E_n$ to E_n.

†The components of $\Omega^\infty E$ are all homotopy equivalent, because $\pi_0(\Omega^\infty E)$ is a group.

The cohomology groups of $\Omega^\infty E$ are usually a lot harder to calculate than the cohomology groups of E, except if one takes cohomology with rational coefficients where the relationship can be described explicitly, as follows:

Given a graded $\mathbb{Q}$-vector space $P^* = \{P^k \mid k > 0\}$, let $A(P^*)$ be the free, graded commutative algebra generated by P^*. It is a polynomial algebra if P^* is concentrated in even degrees, an exterior algebra if P^* is concentrated in odd degrees, and a tensor product of the two in general. A graded basis for P^* serves as multiplicative generators for $A(P^*)$. We give $A(P^*)$ a graded Hopf algebra structure by requiring that P^* be the vector space of primitive elements, so that $A(P^*)$ is primitively generated. The general theory of graded Hopf algebras [55] implies

Theorem 2.10. *There is an isomorphism of Hopf algebras,*

$$H^*(\Omega_0^\infty E;\mathbb{Q}) \cong A(H^{*>0}(E;\mathbb{Q})).$$

For $E = \mathbb{C}P_{-1}^\infty$, the Thom isomorphism shows that $H^*(\mathbb{C}P_{-1}^\infty;\mathbb{Z})$ has one $\mathbb{Z}$ in each even dimension ≥ -2, and hence by the theorem above that

$$H^*(\Omega_0^\infty \mathbb{C}P_{-1}^\infty;\mathbb{Q}) = \mathbb{Q}[\kappa_1,\kappa_2,\ldots], \quad \deg\kappa_i = 2i.$$

In view of Theorem 2.8, this proves Corollary 1.3.

The κ_i are integral cohomology classes, namely the image under the cohomology suspension of generators of $H^*(\mathbb{C}P_{-1}^\infty,\mathbb{Z})$. They correspond to the cohomology classes defined in (1.3), cf. [23]. The main result of [23] is the following theorem about their divisibility in the integral lattice of $H^*(B\Gamma_\infty;\mathbb{Q})$,

$$H^*_{\mathrm{free}}(B\Gamma_\infty;\mathbb{Z}) = H^*(B\Gamma_\infty;\mathbb{Z})/\mathrm{Torsion}.$$

Theorem 2.11 ([23]). *Let D_i be the maximal divisor of κ_i in the integral lattice. It is given by the formulas*

$$D_{2i} = 2 \quad \textit{and} \quad D_{2i-1} = \mathrm{denom}(B_i/2i)$$

with B_i equal to the i'th Bernoulli number.

The maximal divisibility of κ_{2i-1} is what could be expected from the Riemann–Roch theorem [57], [56]. The integral cohomology of $\Omega^\infty\mathbb{C}P_{-1}^\infty$, and thus of $B\Gamma_\infty$, contains a wealth of torsion classes of all orders. This follows from [20] which completely calculates $H^*(B\Gamma_\infty;\mathbb{F}_p)$.

The action of the mapping class group on the first homology group of the underlying surface defines a symplectic representation with kernel equal to the Torelli group. In infinite genus, we get a fibration

$$BT_{\infty,1} \xrightarrow{j} B\Gamma_{\infty,1} \xrightarrow{\pi} B\,\mathrm{SP}_\infty(\mathbb{Z}). \tag{2.8}$$

The rational cohomology ring of $B\,\mathrm{SP}_\infty(\mathbb{Z})$ is a polynomial algebra on $(4i-2)$-dimensional generators that map to non-zero multiples of the κ_{2i-1}, cf. [9], [37]. In (2.8) however, the action of $\mathrm{SP}_\infty(\mathbb{Z})$ on $H^*(BT_{\infty,1})$ is highly non-trivial, so one cannot conclude that $j^*(\kappa_{2i}) \neq 0$. Indeed, this is a wide open problem even for κ_2!

3. Auxiliary moduli spaces

This section presents two extensions of the material in §2. Section 3.1 is an account of the automorphism group of a free group, due entirely to S. Galatius [19]. Section 3.2 presents a topological variant of the Gromov–Witten moduli space of pseudo-holomorphic curves in a background, following the joint work with R. Cohen from [12].

3.1. The moduli space of graphs. Let Aut_n denote the automorphism group of a free group on n letters and Out_n its quotient of outer automorphisms. Using 3-manifold techniques, Hatcher [29] proved that the homomorphisms

$$H_k(B\,\mathrm{Out}_n;\mathbb{Z}) \leftarrow H_k(B\,\mathrm{Aut}_n;\mathbb{Z}) \rightarrow H_k(B\,\mathrm{Aut}_{n+1};\mathbb{Z}) \tag{3.1}$$

are isomorphisms in a range that increases with n. See also [31]. The limit Aut_∞ has perfect commutator subgroup (of index 2), and

$$\mathbb{Z}\times B\,\mathrm{Aut}_\infty^+ \xrightarrow{\simeq} \Omega B\big(\coprod^n B\,\mathrm{Aut}_n\big).$$

Theorem 3.1 ([19]). *The space $\mathbb{Z}\times B\,\mathrm{Aut}_\infty^+$ is homotopy equivalent to the infinite loop space $\Omega^\infty\mathbb{S}$ of the sphere spectrum.*

We remember that $\Omega^\infty\mathbb{S} = \Omega^\infty S^\infty$ is the homotopy colimit of $\Omega^n S^n$ as $n\to\infty$. Its homotopy groups are the stable homotopy groups of spheres; they are finite except for the group of components [67]. The "standard conjecture" in this case:

$$H^k(B\,\mathrm{Aut}_\infty;\mathbb{Q}) = 0 \quad \text{for } k>0, \tag{3.2}$$

is therefore an immediate consequence of Theorem 3.1.

In spirit, the proof of Theorem 3.1 is analogous to the proof of Theorem 2.8: Graphs are 1-dimensional manifolds with singularities. Below, I shall outline the similarities and the new ideas required to prove Theorem 3.1.

Let U be an open set of $\mathbb{R}^{n+1}$. A graph Y in U is a closed subset with the following property. Each $p\in U$ admits an open neighborhood U_p such that one of the following three cases occurs:

(i) $Y\cap U_p=\emptyset$,

(ii) $Y\cap U_p$ is the image of a smooth embedding $(-\varepsilon,\varepsilon)\hookrightarrow U_p$,

(iii) $Y\cap U_p$ is the image of a continuous embedding of the one point union $\bigvee^k[0,\infty)$, $k\geq 3$; the embedding is smooth on each branch and has transverse intersection at the branch point.

The set $\Phi(U)$ of all graphs in U is topologized in a way that allows the (non-loop) edges to shrink to a vertex and allows graphs to be pushed to infinity. The topology on $\Phi(U)$ is similar to the topology on the space $\mathcal{T}_d(\mathbb{R}^{n+d}$ from § 2.3. More precisely, Φ is a space-valued sheaf on the category of open sets of $\mathbb{R}^{n+1}$ and their embeddings. It is an equivariant, continuous sheaf on $\mathbb{R}^{n+1}$ in the terminology of [25, §2.2]. In particular, the restriction $\Phi(\mathbb{R}^{n+1}) \to \Phi(B_\varepsilon(0))$ onto an open ε-ball is a homotopy equivalence.

Graphs in $\mathbb{R}^{n+1}$ give rise to the embedded cobordism category $\mathcal{G}(\mathbb{R}^{n+1})$. Its objects consist of a finite number of points in a slice $\{a\} \times \mathbb{R}^n$. A morphism is a "graph with legs (or leaves)" embedded in a strip $[a_0, a_1] \times \mathbb{R}^n$ with the legs meeting the walls orthogonally.

The restriction of a morphism to the open strip is an element of $\Phi((a_0, a_1) \times \mathbb{R}^n)$. Define

$$G(\mathbb{R}^{n+1}) = \{Y \in \Phi(\mathbb{R}^{n+1}) \mid Y \subset \mathbb{R} \times D^n\}.$$

The proofs of (2.4a) and (2.4b) can be adapted to graphs and show

$$B\mathcal{G}(\mathbb{R}^{n+1}) \simeq G(\mathbb{R}^{n+1}). \tag{3.3}$$

Transversality and Phillips' submersion theorem, used in the proof of Theorem 2.8, requires tangent spaces. This approach is not available for graphs, but instead we have Gromov's general h-principle from [25].

Galatius proves that the sheaf Φ is *microflexible*, and concludes from [25, §2.2] that there is a homotopy equivalence

$$G(\mathbb{R}^{n+1}) \simeq \Omega^n \Phi(\mathbb{R}^{n+1}). \tag{3.4}$$

The collection $\Phi(\mathbb{R}^n)$, with the empty graph as basepoint, forms a spectrum $\mathbf{\Phi}$. The structure maps are induced from the map

$$\mathbb{R} \times \Phi(\mathbb{R}^n) \to \Phi(\mathbb{R}^{n+1})$$

that sends (t, Y) to $\{t\} \times Y \subset \mathbb{R} \times \mathbb{R}^n$. It factors over $\Sigma\Phi(\mathbb{R}^n)$. Let $\mathcal{G} = \mathcal{G}(\mathbb{R}^{\infty+1})$ be the union of the $\mathcal{G}(\mathbb{R}^{n+1})$. In the limit over n, (3.3) and (3.4) yield homotopy equivalences

$$B\mathcal{G} \simeq G(\mathbb{R}^{\infty+1}) \simeq \Omega^{\infty-1}\mathbf{\Phi}. \tag{3.5}$$

Theorem 3.2 ([19]). *The spectrum $\mathbf{\Phi}$ is homotopy equivalent to the sphere spectrum $\mathbb{S}$. In particular $\Omega B\mathcal{G} \simeq \Omega^\infty S^\infty$.*

The proof uses the Handel's theorem from [26] that the space $\exp(X)$ of finite subsets of a connected space X is contractible.

Remark 3.3. One of the advantages of Gromov's h-principle over Phillips' submersion theorem, even in the case of manifolds, is that it permits unstable information.

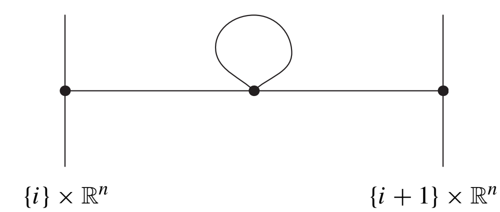
Let $\mathcal{C}_d(\mathbb{R}^{n+d})$ be the category of d-dimensional cobordisms embedded in $\mathbb{R}^{n+d}$ for fixed n. Its classifying space can be identified as

$$B\mathcal{C}_d(\mathbb{R}^{n+d}) \simeq \Omega^{n+d-1}\Psi_d(\mathbb{R}^{n+d}),$$

where $\Psi(\mathbb{R}^{n+d})$ is a space of d-dimensional submanifolds in $\mathbb{R}^{n+d}$, with a topology similar to the above topology on $\Phi(\mathbb{R}^{n+1})$. Theorem 3.2 is replaced by $\Psi_d(\mathbb{R}^{n+d}) \simeq \mathrm{Th}(U_{d,n}^{\perp})$, and we get the following unstable version of Theorem 2.4, valid for any $d \geq 0$ and $n \geq 0$:

$$B\mathcal{C}_d(\mathbb{R}^{n+d}) \simeq \Omega^{n+d-1}\,\mathrm{Th}(U_{d,n}^{\perp}).$$

The final step is to prove that $\Omega B\mathcal{G}$ is homotopy equivalent to $\mathbb{Z} \times B\,\mathrm{Aut}_{\infty}^{+}$. This is very similar to the proof of Theorem 2.7 and Theorem 2.8 in the case of the mapping class group. The contractibility of "outer space" [13], [37, ch. 8] identifies $B\,\mathrm{Aut}_n$ and $B\,\mathrm{Out}_n$ as components of morphism spaces of $\mathcal{G}^{\mathrm{res}}$. Instead of Harer's stability theorem, one uses the generalization [30], [31] of the homology isomorphisms (3.1). The element β_i needed to stabilize the morphism space is the graph



It is an obvious problem to generalize the above to other situations of manifolds with singularities, for example to the case of orbifolds.

3.2. Surfaces in a background. Fix a background space X. What is the stable homology type of the moduli space of pairs (Σ, f) of a Riemann surface Σ, and a continuous map $f\colon \Sigma \to X$?

Let $\mathrm{Emb}_{g,b}^{\infty}$ denote the space of embeddings of a fixed differentiable surface $F = F_{g,b}$ into the strip $[0,1] \times \mathbb{R}^{\infty+1}$ with boundary circles mapped to $\{0,1\} \times \mathbb{R}^{\infty+1}$ when $b > 0$. The moduli space in question can be displayed as the orbit space

$$\mathcal{S}_{g,b}(X) = \mathrm{Emb}_{g,b}^{\infty} \times_{\mathrm{Diff}(F,\partial)} \mathrm{Map}(F, X)$$

of the free $\mathrm{Diff}(F,\partial)$ action on the Cartesian product of $\mathrm{Emb}_{g,b}^{\infty}$ and the space $\mathrm{Map}(F,X)$ of continuous mappings of F into X.

It fibres over the free loop space of X^b, $\pi\colon \mathcal{S}_{g,b}(X) \to LX^b$. The space LX is connected when X is simply connected, and in this case the fibres of π are all homotopy equivalent to the space

$$\mathcal{S}_{g,b}(X; x_0) = \mathrm{Emb}_{g,b}^{\infty} \times_{\mathrm{Diff}(F;\partial F)} \mathrm{Map}((F;\partial F),(X;x_0)).$$

Theorem 3.4 ([12]). *For simply connected X and $b > 0$, the maps*

$$\mathcal{S}_{g,b-1}(X; x_0) \leftarrow \mathcal{S}_{g,b}(X; x_0) \rightarrow \mathcal{S}_{g+1,b}(X; x_0)$$

induce isomorphisms in integral homology in degrees less than or equal to $g/2 - 2$.

Theorem 3.4 might appear a surprise from the following viewpoint: In the fibration

$$\mathrm{Map}((F, \partial F), (X; x_0)) \rightarrow \mathcal{S}_{g,b}(X; x_0) \rightarrow B\,\mathrm{Diff}(F; \partial F),$$

the homology of the base is independent of the genus and of the number of boundary components in a range, whereas the fibre grows in size with g and b. This would seem to prevent stabilization. The explanation is, however, that the fundamental group $\Gamma_{g,b} = \pi_1 B\,\mathrm{Diff}(F; \partial F)$ acts very non-trivially on the homology of the fibre.

The proof of Theorem 3.4 adapts and generalizes ideas from [39] about stability with twisted coefficients. The $\Gamma(F)$-module of twisted coefficients is

$$V_r(F) = H_r(\mathrm{Map}((F; \partial F), (X; x_0)); \mathbb{Z}).$$

It is of "finite degree" [40], [39], precisely when X is simply connected. This explains the unfortunate assumption in Theorem 3.4 that X be simply connected.

Theorem 3.5. *For simply connected X, there is a map*

$$\alpha_X : \mathbb{Z} \times \mathcal{S}_{\infty,b}(X; x_0) \rightarrow \Omega^\infty(\mathbb{C}P^\infty_{-1} \wedge X_+)$$

which induces isomorphism on integral homology.

Given Theorem 3.4, the proof of Theorem 3.5 can be deduced from section 7 of [47]. Alternatively, one can adopt the strategy of §2 above. The categories $\mathcal{C}_d$ and $\mathcal{C}_d^{\mathrm{red}}$ are replaced by the categories $\mathcal{C}_d(X)$ and $\mathcal{C}_d^{\mathrm{red}}(X)$ consisting of embedded cobordisms together with a map from the cobordism into X. Theorem 2.4 and Theorem 2.7 are valid for these categories,

$$B\mathcal{C}_d(X) \simeq B\mathcal{C}_d^{\mathrm{red}}(X) \simeq \Omega^{\infty-1}(\mathbb{G}_{-d} \wedge X_+). \tag{3.6}$$

For $d = 2$, we can apply Lemma 2.6 to complete the proof of Theorem 3.5.

For applications of Theorem 3.5, e.g. in string topology, one needs versions with marked points on the underlying surface. There are two cases: one in which the marked points are allowed to be permuted by the diffeomorphisms, and one in which the marked points are kept fixed. The first case is relevant to open-closed strings [3].

Here are some details. Given a surface $F^s = F^s_{g,b}$ with s interior marked points, let $\mathrm{Diff}(F^{(s)}; \partial F^{(s)})$ be the group of oriented diffeomorphisms that keeps ∂F^s pointwise fixed and permute the marked points. The associated moduli space,

$$\mathcal{S}^{(s)}_{g,b}(X; x_0) = \mathrm{Emb}^\infty_{g,b} \times_{\mathrm{Diff}(F^{(s)}; \partial F^{(s)})} \mathrm{Map}((F^s; \partial F^s); (X; x_0)),$$

fits into a fibration

$$\pi : \mathcal{S}^{(s)}_{g,b}(X; x_0) \to E\Sigma_s \times_{\Sigma_s} (X \times \mathbb{C}P^\infty)^s$$

with fibre $\mathcal{S}_{g,b+s}(X; x_0)$, cf. (1.4). Here Σ_s denotes the permutation group. There are stabilization maps

$$\mathcal{S}^{(s)}_{g,b}(X; x_0) \to \mathcal{S}^{(s+1)}_{g+1,b}(X; x_0) \tag{3.7}$$

that add $F^1_{1,2}$, a torus with two boundary circles and one marked point, to one of the boundary circles.

Infinite loop space theory tells us that the homotopy colimit or telescope of $E\Sigma_s \times_{\Sigma_s} (X \times \mathbb{C}P^\infty)^s$ as $s \to \infty$ is homology equivalent to $\Omega^\infty(\mathbb{S} \wedge (X \times \mathbb{C}P^\infty)_+) = \Omega^\infty S^\infty(X \times \mathbb{C}P^\infty_+)$.

For the moduli space $\mathcal{S}^s_{g,b}(X; x_0)$, where the marked points are kept fixed by the diffeomorphism group, the space $E\Sigma_s \times_{\Sigma_s} (X \times \mathbb{C}P^\infty)^s$ is replaced by $(X \times \mathbb{C}P^\infty)^s$, and $\Omega^\infty S^\infty(X \times \mathbb{C}P^\infty_+)$ by $\Omega S(X \times \mathbb{C}P^\infty_+)$, cf. [18].

Addendum 3.6. There are homology equivalences

(i) $\mathbb{Z} \times \mathbb{Z} \times \mathcal{S}^{(\infty)}_{\infty,b}(X; x_0) \to \Omega^\infty(\mathbb{C}P^\infty_{-1} \wedge X_+) \times \Omega^\infty S^\infty(X \times \mathbb{C}P^\infty_+)$,

(ii) $\mathbb{Z} \times \mathbb{Z} \times \mathcal{S}^{\infty}_{\infty,b}(X; x_0) \to \Omega^\infty(\mathbb{C}P^\infty_{-1} \wedge X_+) \times \Omega S(X \times \mathbb{C}P^\infty_+)$.

It was the Gromov–Witten moduli space and the string topology spaces of M. Chas and D. Sullivan [10], [68] that inspired us to consider the moduli spaces of Theorem 3.5 and Addendum 3.6. While our results are nice in their own right, the question remains if they could be useful for a better understanding of Gromov–Witten invariants and string topology.

The most obvious starting point would be to give a homotopical interpretation of the Batalin–Vilkovsky structure on the chain groups of $\coprod \mathcal{S}_g(M)$, [68], somewhat similar to the homotopical definition in [11] of the Chas–Sullivan loop product.

4. Algebraic K-theory and trace invariants

Quillen's algebraic K-theory space $K(R)$ of a ring R [62], and Waldhausen's algebraic K-theory $A(X)$ of a topological space X [74] are two special cases of algebraic K-theory of "brave new rings". A brave new ring is a ring spectrum whose category of modules fits the axiomatic framework of K-theory laid down in [76]. There are several good choices for the category of brave new rings; one such is Bökstedt's Functors with Smash Products, a closely related one is the more convenient category of symmetric orthogonal spectra [49], [48]. In this category, both K-theory and its companion $\mathrm{TC}(-; p)$ (the topological cyclic homology at p) works well, and one has a good construction of the cyclotomic trace

$$\mathrm{tr}_p \colon K(E) \to \mathrm{TC}(E; p). \tag{4.1}$$

The topological cyclic homology and the trace map tr_p was introduced in joint work with M. Bökstedt and W.-C. Hsing in order to give information about the p-adic homotopy type of $K(E)$, cf. [7].

The Eilenberg–MacLane spectrum $H(R)$ of a ring R is a brave new ring; its K-theory is equivalent to Quillen's original $K(R)$. The K-theory of the brave new ring $\mathbb{S} \wedge \Omega X_+$ is Waldhausen's $A(X)$.

The product over all primes of the p-adic completion of $\mathrm{TC}(E;\, p)$ can be combined with a version of negative cyclic homology for $E \otimes \mathbb{Q}$ to define $\mathrm{TC}(E)$, [24]. There is a trace map from $K(E)$ to $\mathrm{TC}(E)$ whose p-adic completion is the p-adic completion of tr_p, and a map from $\mathrm{TC}(H(R))$ to the usual negative cyclic homology of $R \otimes \mathbb{Q}$.

4.1. $A(X)$ and diffeomorphisms. I begin with Waldhausen's definition of $A(\mathrm{pt})$. Let S_q be the category of length q "flags" of based, finite CW-complexes. An object consists of a string

$$X_1 \rightarrowtail X_2 \rightarrowtail \cdots \rightarrowtail X_q\ , \tag{4.2}$$

where the arrows are based cellular inclusions ($S_0 = *$). The morphisms in S_q are the based homotopy equivalences of flags. There are face operators

$$d_i \colon S_q \to S_{q-1}, \quad i = 0, \ldots, q,$$

where d_0 divides out X_1 in that it replaces (4.2) with the flag

$$X_2/X_1 \rightarrowtail \cdots \rightarrowtail X_q/X_1\ ,$$

and d_i forgets X_i, when $i > 0$. This makes $S_\bullet$ into a simplicial category.

The classifying spaces BS_q with the induced d_i form a simplicial space $BS_\bullet$, and $A(\mathrm{pt})$ is the loop space of its topological realization,

$$A(\mathrm{pt}) = \Omega|BS_\bullet| = \Omega\big(\coprod \Delta^k \times BS_k/\!\sim\big). \tag{4.3}$$

(The definition of $A(X)$ is similar; S_q is replaced with the set of flags

$$X \rightarrowtail X_1 \rightarrowtail \cdots \rightarrowtail X_q \longrightarrow X,$$

whose composite is the identity map of X, and with $(X_i;\, X)$ a finite relative CW-complex). By definition,

$$BS_1 = \coprod B\,\mathrm{Aut}(Y)$$

where Y runs over finite CW-complexes and $\mathrm{Aut}(Y)$ denotes the monoid of pointed self-homotopy equivalences. The map from $\Delta^1 \times BS_1$ into $|BS_\bullet|$ induces a map

$$B\,\mathrm{Aut}(Y) \longrightarrow A(\mathrm{pt}), \tag{4.4}$$

and one may view $A(\mathrm{pt})$ as a kind of moduli space of finite CW-complexes.

Given the homotopy theoretic nature of the construction of $A(\mathrm{pt})$ and more generally of $A(X)$, the following theorem from [74] and [75] seems miraculous.

Theorem 4.1 ([74]). *There are homotopy equivalences*

(i) $A(\mathrm{pt}) \simeq \operatorname{hocolim} \Omega^n(\mathrm{Top}_{n+1} / \mathrm{Top}_n)$,

(ii) $A(X) \simeq \Omega^\infty S^\infty(X_+) \times \mathrm{Wh}^{\mathrm{Diff}}(X)$,

(iii) $\Omega^2 \mathrm{Wh}^{\mathrm{Diff}}(X) \simeq \operatorname{hocolim}_k \mathrm{Diff}(X \times I^{k+1}; X \times I^k)$, *provided X is a smooth manifold.*

In (i), Top_{n+1} denotes the topological group of homeomorphisms of $\mathbb{R}^{n+1}$, and $\mathrm{Top}_{n+1} / \mathrm{Top}_n$ is the associated homogenous space. The right-hand side of (iii) is the homotopy colimit of the space of diffeomorphisms of $X \times I^{k+1}$ that induces the identity on the submanifold $X \times I^k$ of the boundary, the so called stable pseudo-isotopy space of X. There is a stability range for pseudo-isotopy spaces in terms of $\dim(X)$ due to K. Igusa [36], namely

$$\mathrm{Diff}(X \times I; X \times 0) \to \operatorname{hocolim}_k \mathrm{Diff}(X \times I^{k+1}, X \times I^k) \tag{4.5}$$

induces isomorphism on homotopy groups in degree i if $3i < \dim X - 7$.

Theorem 4.1 is the conclusion of a long development in geometric topology. See the references in [75], where its proof was sketched out. The final details are due to appear in [77].

4.2. Topological cyclic homology. There is a trace map, due to K. Dennis, from $K(R)$ into a space $\mathrm{HH}(R)$, whose homotopy groups are the Hochschild homology groups of R with coefficients in the bimodule R itself, cf. [43, 8.4]. This was generalized to brave new rings by M. Bökstedt who constructed the topological Hochschild homology space $\mathrm{THH}(E)$ and a trace map from $K(E)$ into it. $\mathrm{THH}(E)$ is a cyclic space in the sense of Connes [43], so it comes equipped with a continuous action of the circle group $\mathbb{T}$. (Actually, $\mathrm{THH}(E)$ is a brave new ring with a $\mathbb{T}$-action.) Topological cyclic homology $\mathrm{TC}(E; p)$ is made out of the fixed sets $\mathrm{THH}(E)^{C_{p^n}}$ of the cyclic subgroups of $\mathbb{T}$.

The invariant (4.1) is the key tool for our present understanding of $A(X)$ and more generally $K(E)$. Consider a homomorphism $\varphi\colon E \to F$ of brave new rings and the induced diagram

$$\begin{tikzcd} K(E) \arrow[r] \arrow[d] & \mathrm{TC}(E; p) \arrow[d] \\ K(F) \arrow[r] & \mathrm{TC}(F; p). \end{tikzcd} \tag{4.6}$$

The following theorem of B. Dundas contains a basic relationship between K and TC. The theorem was conjectured by T. Goodwillie in [24], and proved in a special case by R. McCarthy in [50].

Theorem 4.2 ([14]). *Suppose $\pi_0 E \to \pi_0 F$ is a surjective ring homomorphism with nilpotent kernel. Then* (4.6) *becomes homotopy Cartesian after p-adic completion.*

I will now turn to the calculation of $\mathrm{TC}(X; p)$, the topological cyclic homology of the brave new ring $\mathbb{S} \wedge \Omega X_+$. There is a commutative diagram

$$\begin{tikzcd} \mathrm{TC}(X; p) \arrow[r] \arrow[d] & \Omega^{\infty-1} S^{\infty}(E\mathbb{T} \times_{\mathbb{T}} LX_+) \arrow[d, "\mathrm{trf}"] \\ \Omega^{\infty} S^{\infty}(LX_+) \arrow[r, "1-\Delta_p"] & \Omega^{\infty} S^{\infty}(LX_+), \end{tikzcd} \tag{4.7}$$

where $\Delta_p \colon LX \to LX$ sends a loop $\lambda(z)$ to $\lambda(z^p)$, $z \in S^1$, and trf is the $\mathbb{T}$-transfer map (a fibrewise Pontryagin–Thom collapse map).

Theorem 4.3 ([7]). *The diagram* (4.7) *becomes homotopy Cartesian after p-adic completion.*

When X is a single point, the theorem reduces to the statement

$$\mathrm{TC}(\mathrm{pt}; p) \simeq_p \Omega^{\infty} S^{\infty} \times \Omega^{\infty-1} \mathbb{C}P^{\infty}_{-1}, \tag{4.8}$$

with $\simeq_p$ indicating that the two sides become homotopy equivalent after p-adic completion. In particular the two sides have the same mod p homotopy groups.

Theorem 4.2, applied to the case in which E is the sphere spectrum and F is the Eilenberg–Maclane spectrum of the integers, tells us that the homotopy fibres of $A(\mathrm{pt}) \to K(\mathbb{Z})$ and $\mathrm{TC}(\mathrm{pt}; p) \to \mathrm{TC}(\mathbb{Z}; p)$ have the same p-adic completions. This reduces the understanding of $A(\mathrm{pt})$ to the understanding of $K(\mathbb{Z})$ and the fibre of $\mathrm{TC}(\mathrm{pt}; p) \to \mathrm{TC}(\mathbb{Z}; p)$. The structure of $\mathrm{TC}(\mathbb{Z}; p)$ is given in [8], [63].

If the ring R is finitely generated as an abelian group, then

$$\mathrm{TC}(R; p) \simeq_p \mathrm{TC}(R \otimes \mathbb{Z}_p; p).$$

The corresponding statement for K-theory is false, so the absolute invariant (4.1) is effective mostly for p-complete rings. There is an extensive theory surrounding the functor $\mathrm{TC}(R; p)$, developed in joint work with L. Hesselholt in [33], [34]. See also [32]. The p-adic completion of $\mathrm{TC}(R; p)$ is explicitly known when R is the ring of integers in a finite field extension of $\mathbb{Q}$, and in this case (4.1) induces an equivalence

$$K(R \otimes \mathbb{Z}_p) \simeq_p \mathrm{TC}(R; p).$$

For global rings, the motivic theory is the basic tool for calculations of $K(R)$; see F. Morel's article in these proceedings.

In view of Theorem 4.2 and Theorem 4.3, it is of obvious interest to examine the map from $\mathrm{TC}(X; p)$ to $\mathrm{TC}(\mathbb{Z}[\pi_1 X]; p)$, but essentially nothing is known about this problem. Two special cases are of particular interest. The case $X = \mathrm{pt}$ is required for the homotopical control of the fibre of $A(\mathrm{pt}) \to K(\mathbb{Z})$. The case $X = S^1$ is, by theorems of T. Farrell and L. Jones, related to the homotopical structure of the group of homeomorphisms of negatively curved closed manifolds, cf. [45].

Let $\mathbb{K}$ be the periodic spectrum whose $2n$'th space is $\mathbb{Z} \times BU$ and with structure maps induced from Bott periodicity. It is believed that $A(\mathbb{K})$ will be important in the study of field theories, so one would like to understand the homotopy type of $\mathrm{TC}(\mathbb{K}; p)$. There are helpful partial results from [2], [1], but the problem seems to be a very difficult one.

Finally, and maybe most important, there are reasons to believe that the moduli space of Riemann surfaces is related to $\mathrm{TC}(\mathrm{pt}; p)$, possibly via field theories. The spectrum $\mathbb{C}P^{\infty}_{-1}$ occurs in both theories. It is a challenge to understand why.

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Department of Mathematical Sciences, University of Aarhus, Denmark.
E-mail: imadsen@imf.au.dk