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The stable mapping class group and $Q(\mathbb{C}P^\infty_+)$

Ib Madsen[1], **Ulrike Tillmann**[2,⋆]

[1] Matematisk Institut, Aarhus Universitet, 8000 Aarhus C, Denmark
(e-mail: imadsen@imf.au.dk)
[2] Mathematical Institute, 24–29 St. Giles Street, Oxford OX1 3LB, UK
(e-mail: tillmann@maths.ox.ac.uk)

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Abstract. In [T2] it was shown that the classifying space of the stable mapping class groups after plus construction $\mathbb{Z} \times B\Gamma^+_\infty$ has an infinite loop space structure. This result and the tools developed in [BM] to analyse transfer maps, are used here to show the following splitting theorem. Let $\Sigma^\infty(\mathbb{C}P^\infty_+)^\wedge_p \simeq E_0 \vee \cdots \vee E_{p-2}$ be the "Adams-splitting" of the p-completed suspension spectrum of $\mathbb{C}P^\infty_+$. Then for some infinite loop space W_p,

$$\left(\mathbb{Z} \times B\Gamma^+_\infty\right)^\wedge_p \simeq \Omega^\infty(E_0) \times \cdots \times \Omega^\infty(E_{p-3}) \times W_p$$

where $\Omega^\infty E_i$ denotes the infinite loop space associated to the spectrum E_i. The homology of $\Omega^\infty E_i$ is known, and as a corollary one obtains large families of torsion classes in the homology of the stable mapping class group. This splitting also detects all the Miller-Morita-Mumford classes. Our results suggest a homotopy theoretic refinement of the Mumford conjecture. The above p-adic splitting uses a certain infinite loop map

$$\alpha_\infty : \mathbb{Z} \times B\Gamma^+_\infty \longrightarrow \Omega^\infty \mathbb{C}P^\infty_{-1}$$

that induces an isomorphims in rational cohomology precisely if the Mumford conjecture is true. We suggest that α_∞ might be a homotopy equivalence.

1. Introduction and statement of theorems

1.1. Mumford's conjecture

For an oriented surface F, we let $\mathrm{Diff}(F; \partial)$ denote the topological group of *orientation preserving* diffeomorphisms that keep the boundary ∂F point-

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wise fixed when $\partial F \neq \emptyset$. The components of $\mathrm{Diff}(F;\partial)$ are contractible when the genus of F is greater than one [EE], [ES], so $B\mathrm{Diff}(F;\partial) \simeq B\Gamma(F)$ where $\Gamma(F) = \pi_0\mathrm{Diff}(F;\partial)$ is the mapping class group of F. $B\Gamma(F;\partial)$ also has the same homotopy type as the moduli space of Riemann surfaces of type F if $\partial F \neq \emptyset$ and the same rational homotopy type if F is closed.

In [Mu] Mumford conjectured that $H^*(B\Gamma(F);\mathbb{Q})$ is, in low dimensions compared to the genus, a polynomial algebra on classes κ_i of degree $2i$. To construct κ_i, let $F \to E \to B$ be a smooth oriented surface bundle, and T^vE the tangent bundle along the fibres. This is an oriented 2-plane bundle over E, i.e. a complex line bundle. Let

$$\int_F : H^{2i+2}(E) \to H^{2i}(B)$$

be the "integration along the fibres" map. One defines characteristic classes

(1.1) $$\kappa_i = \int_F c_1(T^vE)^{i+1} \quad \in H^{2i}(B).$$

In the universal situation:

(1.2)
$$F \longrightarrow E(F) \xrightarrow{\pi_F} B\mathrm{Diff}(F;\partial), \quad E(F) = E\mathrm{Diff}(F;\partial) \times_{\mathrm{Diff}(F;\partial)} F,$$

where F is some compact surface, one has

$$T^vE(F) = E\mathrm{Diff}(F;\partial) \times_{\mathrm{Diff}(F;\partial)} TF,$$

and gets classes $\kappa_i(F) \in H^{2i}(B\mathrm{Diff}(F;\partial))$.

It is convenient to let the genus of the surfaces go to infinity. Let $F_{g,1+1}$ denote a genus g surface with two boundary components. One may glue to one of the boundary components a torus with two boundary circles to get an inclusion into $F_{g+1,1+1}$, and hence a map

$$B\mathrm{Diff}(F_{g,1+1};\partial) \longrightarrow B\mathrm{Diff}(F_{g+1,1+1};\partial).$$

The mapping class group $\Gamma_{g,1+1} = \Gamma(F_{g,1+1})$ is perfect for $g \geq 3$ [P]. So an application of Quillen's plus construction yields simply connected spaces with unchanged homology and cohomology. The maps

(1.3) $$B\Gamma^+_{g,1+1} \longrightarrow B\Gamma^+_{g+1,1+1}, \quad B\Gamma^+_{g,1+1} \longrightarrow B\Gamma^+_g$$

are thus $[g/2]$-connected, respectively $[(g-2)/2]$-connected by the homology stability theorems of Harer and Ivanov [H2], [I]. The homotopy direct limit of the maps in (1.3) as $g \to \infty$ is denoted $B\Gamma^+_\infty$. Its homology is the stable homology of the mapping class group. Note that from (1.3) *Mumford's conjecture* takes the form

(1.4) $$H^*\big(B\Gamma^+_\infty;\mathbb{Q}\big) = \mathbb{Q}[\kappa_1,\kappa_2,\dots].$$

1.2. Refinement of Mumford's conjecture

The integration along fibre used to define the classes κ_i can be given a more homotopical formulation. To this end let us return to the smooth fibre bundle $F \to E \xrightarrow{\pi} B$ and suppose for simplicity that F is a *closed surface*. Choose a smooth embedding of E in some Euclidean space $\mathbb{R}^k$. The normal bundle $N^v E$ of the resulting embedding $E \subset B \times \mathbb{R}^k$ is the "normal bundle along the fibres",

$$T^v E \oplus N^v E \simeq E \times \mathbb{R}^k.$$

Collapsing the complement of a tube around E in $B \times \mathbb{R}^k$ induces a map

$$B_+ \wedge S^k \longrightarrow \mathrm{Th}(N^v E) \tag{1.5}$$

into its Thom space (one point compactification of $N^v E$ assuming B is compact). The induced map in cohomology composed with the Thom isomorphism is the integration homomorphism $\int_F$. The composition of (1.5) with the inclusion

$$\mathrm{Th}(N^v E) \xrightarrow{\omega} \mathrm{Th}(T^v E \oplus N^v E)$$

defines a map from $B_+ \wedge S^k$ to $E_+ \wedge S^k$ and hence as $k \to \infty$ the map

$$\mathrm{trf}(\pi) : B \longrightarrow Q(E_+), \qquad Q = \Omega^\infty S^\infty. \tag{1.6}$$

This is the Becker-Gottlieb transfer map, [BG].

Consider the canonical line bundle L_s over the complex projective space $\mathbb{C}P^s$ and its complementary $\mathbb{C}^s$-bundle $-L_s$. We use the notation

$$\Omega^\infty \mathbb{C}P^\infty_{-1} = \operatorname{colim} \Omega^{2s+2}\mathrm{Th}(-L_s),$$
$$Q(S^{-1}) = \operatorname{colim} \Omega^{s+1} S^s.$$

There is a fibration, [R]

$$\Omega^\infty \mathbb{C}P^\infty_{-1} \xrightarrow{\omega_\infty} Q\big(BS^1_+\big) \xrightarrow{\partial} Q(S^{-1}). \tag{1.7}$$

Here and in the rest of the paper we prefer to write BS^1 instead of $\mathbb{C}P^\infty$. The map ∂ is the so called circle transfer map. The oriented 2-plane bundle $T^v E$ is classified by L_s for s sufficiently large, giving a map from $\mathrm{Th}(N^v E)$ to $\mathrm{Th}(-L_s)$ and universally maps

$$\alpha_F : B\mathrm{Diff}(F) \to \Omega^\infty \mathbb{C}P^\infty_{-1}, \qquad \alpha_\infty : \mathbb{Z} \times B\Gamma^+_\infty \longrightarrow \Omega^\infty \mathbb{C}P^\infty_{-1}.\text{[1]}$$

The rest of the paper revolves around the following extension of (1.4).

[1] The indicated construction of α_F and α_∞ only gives well-defined homotopy classes on compact subcomplexes of $B\mathrm{Diff}(F)$ and $\mathbb{Z} \times B\Gamma^+_\infty$ due to possible "$\lim^1$-problems". Section 2 below however gives a different construction that is indeed well-defined on all of $B\mathrm{Diff}(F)$ and $\mathbb{Z} \times B\Gamma^+_\infty$, see Theorem 2.6 and the discussion following (2.12).

Conjecture 1.1. The map $\alpha_\infty : \mathbb{Z} \times B\Gamma_\infty^+ \to \Omega^\infty \mathbb{C}P_{-1}^\infty$ is a homotopy equivalence.

We note that the conjecture does indeed extend Mumford's conjecture (1.4). The map ω_∞ of (1.7) is a rational equivalence, and so is the map

$$\hat{L} : Q\big(BS_+^1\big) = Q(S^0) \times Q(BS^1) \longrightarrow \mathbb{Z} \times B\mathrm{U} \tag{1.8}$$

that is the degree map on the first factor and the reduced canonical line bundle on the second factor, extended over $Q(BS^1)$ via Bott periodicity (see below). There results an isomorphims in cohomology of say 0-components

$$H^*\big(\Omega_0^\infty \mathbb{C}P_{-1}^\infty; \mathbb{Q}\big) = H^*\big(Q_0\big(BS_+^1\big); \mathbb{Q}\big) = H^*(B\mathrm{U}; \mathbb{Q}).$$

Moreover the classes κ_i are related to the above maps by the formula

$$\kappa_i = \alpha_\infty^* \omega_\infty^* \hat{L}^*(i!ch_i) \tag{1.9}$$

where ch_i denotes the i-th component of the Chern character, cf. Sect. 4.2.

The conjecture can be given a more geometric flavor by applying the oriented bordism functor Ω_*^{SO} to α_∞, cf. [CF]. The n-th group $\Omega_n^{\mathrm{SO}}(B\Gamma_\infty)$ consists of bordisms classes of oriented surface bundles $F \to E^{n+2} \to M^n$ with the genus of F large compared to n. By transversality, $\Omega_n^{\mathrm{SO}}(\Omega_0^\infty \mathbb{C}P_{-1}^\infty)$ is the group of cobordism classes of 4-tuples $(E^{n+2}, \psi, M^n, \xi)$ with ξ an oriented $\mathbb{R}^2$-bundle over E and ψ an arbitrary smooth map from E to M subject only to the condition that TE is stably equivalent to $\xi \oplus \psi^* TM$. For any homology theory E, $E_*(X) = E_*(X^+)$. Oriented bordism is a homology theory, so

$$\Omega_*^{\mathrm{SO}}\big(B\Gamma_\infty^+\big) = \Omega_*^{\mathrm{SO}}(B\Gamma_\infty).$$

Moreover $\Omega_*^{\mathrm{SO}}(\alpha_\infty)$ is an isomorphism for all values of $*$ if and only if $H_*(\alpha_\infty)$ is an isomorphism. Hence the conjecture is equivalent to the assertion that each cobordism class $[E^{n+2}, \psi, M^n, \xi]$ contains a "unique" representative with ψ a submersion. Note though that a necessary condition for the bordism class $[E, \psi, M, \xi]$ to contain a surface bundle is that the bundle data can be destabilized up to bordism. One might expect 2-primary obstructions to do so which suggests in the statement of Conjecture 1.1 the spaces should maybe be localized away from two.

The cohomology of $B\Gamma_\infty$ has been calculated for $* \leq 2$ in [P], [H1] and [H3]: $H^1(B\Gamma_\infty) = 0$, $H^2(B\Gamma_\infty) = \mathbb{Z}$, and $H^3(B\Gamma_\infty) = 0$. We will show in Sect. 4.2 that in these dimensions (as well as on connected components) α_∞ induces an isomorphism. Let us finally note the fourth cohomology group of the stable mapping class group is of rank two by [H4], while, cf. (4.5),

$$H^4\big(\Omega_0^\infty \mathbb{C}P_{-1}^\infty\big) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/24. \tag{1.10}$$

Let $\eta : B\Gamma^+_\infty \to B\mathrm{Sp}(\mathbb{R})$ be the map that results from the action of the mapping class group on the first homology of the underlying surface. Since the maximal compact subgroup of the symplectic group $\mathrm{Sp}_{2g}(\mathbb{R})$ is the unitary group U_g, $B\mathrm{Sp}(\mathbb{R}) \simeq B\mathrm{U}$ and η can be considered as a (stable) unitary bundle. Mumford and Morita, see e.g. [Mo; p. 555], prove that

$$\eta^*((2i-1)!\,ch_{2i-1}) = (-1)^i \frac{B_i}{2i}\,\kappa_{2i-1}. \tag{1.11}$$

In Sect. 4.3 below we construct $\eta' : \Omega^\infty_0 \mathbb{C}P^\infty_{-1} \to B\mathrm{U}$ such that (1.11) is satisfied for η replaced by $\eta' \circ \alpha_\infty$, showing that (1.11) is in agreement with Conjecture 1.1.

1.3. Methods and results

By the main result in [T2], the space $\mathbb{Z} \times B\Gamma^+_\infty$ is an infinite loop space in the sense that there exists a sequence of spaces E_n with $E_n = \Omega E_{n+1}$ and

$$\mathbb{Z} \times B\Gamma^+_\infty = E_0.$$

We reprove this in Sect. 2 using a somewhat different model for the space $\mathbb{Z} \times B\Gamma^+_\infty$. By definition $\Omega^\infty \mathbb{C}P^\infty_{-1}$ is also an infinite loop space, and the main purpose of Sect. 2 is to show that the map α_∞ is an infinite loop map. This allows us to use the tools of stable homotopy theory.

In particular, the category of infinite loop spaces and infinite loop maps has three obvious advantages over the category of spaces. First, a map $f : X \to E_0$ into an infinite loop space extends uniquely (up to homotopy) to an infinite loop map $\hat{f}$,

$$\begin{array}{ccc} X & \longrightarrow & Q(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \hat{f}} \\ E_0 & = & E_0. \end{array}$$

Second, an infinite loop map $f : E_0 \to E_0'$ is a homology equivalence if and only if it induces an isomorphism on spectrum homology,

$$H_n^{\mathrm{spec}}(E_0) = \operatorname{colim} H_{n+k}(E_k).$$

In particular, an infinite loop map from $Q(X)$ to $Q(Y)$ is a homology equivalence precisely if the induced map from $H_*(X)$ to $H_*(Y)$ is an isomorphism. Third, any map from X to E_0 factors over Quillen's plus construction (with respect to any perfect subgroup of $\pi_1 X$),

$$[X, E_0] = \left[X^+, E_0\right].$$

In Sect. 3 we produce maps

$$\rho_F : BC_{p^n} \longrightarrow B\mathrm{Diff}(F)$$

by exhibiting a suitable closed surface F equipped with an action of C_{p^n}, and then study $\tau_F \circ \rho_F$ where τ_F is the transfer map trf_{π_F} followed by the map induced by the classifying map of the vertical tangent bundle over $E(F)$, cf. (1.2). This amounts to a study of the Becker-Gottlieb transfer for bundles

$$\pi_n : EC_{p^n} \times_{C_{p^n}} F \longrightarrow BC_{p^n},$$

where we can use results from [BM]: If F admits a non-degenerate vector field X and $S(X)$ denotes its singular set, then π_n contains the covering space

$$EC_{p^n} \times_{C_{p^n}} S(X) \longrightarrow BC_{p^n}$$

and the transfer $\mathrm{trf}(\pi_n)$ is expressible in terms of the covering space transfer, which is easy to calculate.

In order to assemble the maps ρ_F to a map into $\mathbb{Z} \times B\Gamma_\infty^+$ one needs to pass to p-completions in the sense of [BK, Chap. VI]. For an infinite loop space E of finite type, i.e. with finitely generated homotopy groups in each degree, one has

$$\text{(1.12)} \qquad \lim\left[BC_{p^n}, E_p^\wedge\right] = \left[BS^1, E_p^\wedge\right] = [BS^1, E] \otimes \mathbb{Z}_p$$

where $E_p^\wedge$ denotes the p-completion of E and $\mathbb{Z}_p$ is the ring of p-adic integers. This applies to $E = \mathbb{Z} \times B\Gamma_\infty^+$. Using the ρ_F and (1.12) we construct in Theorem 3.6 a map from BS^1 into $(B\Gamma_\infty^+)_p^\wedge$. The infinite loop structure on $B\Gamma_\infty^+$ gives an extension $\tilde{\mu}_p : Q(BS^1) \longrightarrow (B\Gamma_\infty^+)_p^\wedge$. There is also a map from QS^0 into $\mathbb{Z} \times B\Gamma_\infty^+$, namely the unique infinite loop map that extends the embedding $S^0 \subset \mathbb{Z} \times B\Gamma_\infty^+$ which sends the non-base point into $(1, *)$. In all we get an infinite loop map

$$\mu_p : Q\left(BS_+^1\right) \longrightarrow \left(\mathbb{Z} \times B\Gamma_\infty^+\right)_p^\wedge.$$

In Sect. 3 we calculate the composition of the map μ_p and the p-completion of

$$\tau_\infty : \mathbb{Z} \times B\Gamma_\infty^+ \xrightarrow{\alpha_\infty} \Omega^\infty \mathbb{C}P_{-1}^\infty \xrightarrow{\omega_\infty} Q\left(BS_+^1\right).$$

Let

$$\psi^k : BS^1 \longrightarrow (BS^1)_p^\wedge$$

be the map that represents $k \cdot c_1(L)$ in $H^2(BS^1; \mathbb{Z}_p)$. It extends uniquely (up to homotopy) to a self map of $Q(BS^1)_p^\wedge$ which we again denote by ψ^k. Our main result is

Theorem 1.2. *The composition $\tau_\infty \circ \mu_p$ is homotopic to the map*

$$\begin{pmatrix} -2 & * \\ 0 & 1 - g\psi^g \end{pmatrix} : (QS^0 \times Q(BS^1))_p^\wedge \longrightarrow (QS^0 \times Q(BS^1))_p^\wedge,$$

where $g \in \mathbb{Z}_p^\times$ is a topological generator ($g = 3$ if $p = 2$).

The diagonal entry $-2 : QS^0 \to QS^0$ was initially proved in [T3], and is reproved below, see (2.14) and the end of Sect. 3.3. The bottom diagonal entry is identified in Theorem 3.6.

The theorem contains the result of Miller and Morita from [Mi] and [Mo] because $1 - g\psi^g$ is non-zero on each homology group. Indeed as mentioned earlier $Q(BS^1_+)$ is rationally equivalent to $\mathbb{Z} \times B\mathrm{U}$, so $H^*(B\Gamma^+_\infty; \mathbb{Q}_p)$ contains the polynomial algebra $H^*(B\mathrm{U}; \mathbb{Q}_p)$. The same will then be the case with $\mathbb{Q}$ coefficients. But Theorem 1.2 is much stronger.

In Sect. 4 we show that

$$Q\big(BS^1_+\big)^\wedge_p \simeq \Omega^\infty E_0 \times \cdots \times \Omega^\infty E_{p-2}$$

for p-complete infinite loop spaces E_i ($\Omega^\infty E_0 = (QS^0)^\wedge_p \times \Omega^\infty \tilde{E}_0$). The splitting is preserved by Adams operations, so that

$$1 - g\psi^g : \Omega^\infty \tilde{E}_i \longrightarrow \Omega^\infty \tilde{E}_i \qquad (E_i = \tilde{E}_i,\ i > 0)$$

and we calculate the induced map on spectrum homology. The result is that for $p > 2$, $1 - g\psi^g$ is a homotopy equivalence on $\Omega^\infty \tilde{E}_i$ for $i \neq p-2$. This implies the splitting listed in the abstract, and exhibits large families of torsion classes in $H^*(B\Gamma_\infty)$, cf. Corollary 4.5.

Finally we combine with the main result from [MS] which asserts that at odd primes p there exists an infinite loop map

$$l_{-1} : \big(\Omega^\infty_0 \mathbb{C}P^\infty_{-1}\big)^\wedge_p \longrightarrow B\mathrm{U}^\wedge_p$$

which is split surjective in the sense that there exists a map σ_{-1} in the other direction with $l_{-1} \circ \sigma_{-1} \simeq id$. The map σ_{-1} is not an infinite loop map however – it only deloops once.

Theorem 1.3. *For odd primes p, the composition*

$$l_{-1} \circ \alpha_\infty : \big(B\Gamma^+_\infty\big)^\wedge_p \longrightarrow B\mathrm{U}^\wedge_p$$

is split surjective.

This is a $\mathbb{Z}_p$ integral version of the theorem by Miller and Morita: the polynomial algebra $\mathbb{Z}_p[c_1, c_2, \dots]$ is a split summand of $H^*(B\Gamma^+_\infty; \mathbb{Z}_p)$.

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2. The pretransfer as a map of infinite loop spaces

The goal of this section is to define the map

$$\alpha_\infty : \mathbb{Z} \times B\Gamma_\infty^+ \longrightarrow \Omega^\infty \mathbb{C}P_{-1}^\infty$$

of Conjecture 1.1 as a map of infinite loop spaces. Indeed, we will define it as a map of simplicial Γ-spaces (in the sense of [S1]) which represent deloops of the above spaces. The main technical challenge is to construct a model for $\mathbb{Z} \times B\Gamma_\infty^+$ out of surfaces embedded in $\mathbb{R}^\infty$ so that the transfer map can readily be defined. This model also has an infinite loop space structure that is easily seen to be compatible with that of $\Omega^\infty \mathbb{C}P_{-1}^\infty$.

2.1. Unparametrized surfaces in $\mathbb{R}^\infty$

We construct a topological category $\mathcal{Y}$ of unparametrized, oriented, one dimensional manifolds and cobordisms in $\mathbb{R}^\infty = \mathrm{colim}_{n\to\infty}\mathbb{R}^n$.

Objects in $\mathcal{Y}$. Let

$$S_n = S^1 \sqcup \cdots \sqcup S^1, \qquad n - \text{summands}$$

be the disjoint union of n circles and let $\mathrm{Emb}(S_n; \mathbb{R}^\infty)$ be the space of smooth embeddings of S_n in $\mathbb{R}^\infty$ with the usual topology, see e.g. [Hi; p. 35]. The group of orientation preserving diffeomorphisms of S_n is the wreath product of the group $\mathrm{Diff}(S^1)$ of orientation preserving diffeomorphisms of S^1 twisted by the symmetric group on n letters,

$$\mathrm{Diff}(S_n) = \Sigma_n \wr \mathrm{Diff}(S^1) \simeq \Sigma_n \wr \mathrm{SO}(2).$$

It acts freely on $\mathrm{Emb}(S_n; \mathbb{R}^\infty)$ by precomposition

$$(\phi.\alpha)(x) = \alpha(\phi^{-1}(x)),$$

where $\phi \in \mathrm{Diff}(S_n)$, $\alpha \in \mathrm{Emb}(S_n; \mathbb{R}^\infty)$, and $x \in S_n$. We define the objects in $\mathcal{Y}$ to be the orbits under this action,

$$\mathrm{ob}\,\mathcal{Y} = \coprod_{n\geq 0} \mathrm{Emb}\big(S_n; \mathbb{R}^\infty\big)/\mathrm{Diff}(S_n).$$

Given an embedding $\alpha \in \mathrm{Emb}(S_n; \mathbb{R}^\infty)$, we think of the associated object $[\alpha] \in \mathcal{Y}$ as the image $\mathrm{im}(\alpha)$ (with an orientation).

Morphisms in $\mathcal{Y}$. Let F be an abstract oriented smooth cobordism from S_n to S_m. Assume that the incoming and outgoing boundary of F are parametrized and have a collar, i.e. there are identifications

$$\partial_- F = S_n \quad \text{and} \quad \partial_+ F = S_m$$

which extend on an open neighborhood of the boundary to a collar $\mathcal{O}(\partial_- F) = S_n \times [0, \epsilon)$ and $\mathcal{O}(\partial_+ F) = S_m \times (-\epsilon, 0]$ for some $\epsilon > 0$. Let

$$\mathrm{Emb}^\Omega(F; [0, t] \times \mathbb{R}^\infty), \qquad t > 0$$

denote the space of smooth embeddings $h : F \hookrightarrow [0, t] \times \mathbb{R}^\infty$ such that

$$h(\partial_- F) = h(F) \cap \{0\} \times \mathbb{R}^\infty, \qquad h(\partial_+ F) = h(F) \cap \{t\} \times \mathbb{R}^\infty,$$

and the collars are mapped to the standard ϵ_t-collar of $\partial[0, t] \times \mathbb{R}^\infty$ (e.g. with $\epsilon_t = t/100$). More precisely, restricted to the collars of incoming and outgoing boundary,

$$h(x, s) = (f_-(s), \partial_- h(x)), \qquad h(x, s) = (f_+(s), \partial_+ h(x))$$

where $f_- : [0, \epsilon) \to [0, \epsilon_t)$ and $f_+ : (-\epsilon, 0] \to (t-\epsilon_t, t]$ are linear and $\partial_- h$ and $\partial_+ h$ are the restriction of h to $\partial_- F$ and $\partial_+ F$ respectively. Let $\mathrm{Diff}^\Omega(F)$ denote the group of diffeomorphisms ϕ of the cobordism F that restrict on the collar to diffeomorphisms of the form

$$\partial_- \phi \times id_{[0,\epsilon)}, \qquad \partial_+ \phi \times id_{(-\epsilon,0]}.$$

Again, $\partial_- \phi$ and $\partial_+ \phi$ denote the restrictions of ϕ to $\partial_- F$ and $\partial_+ F$. $\mathrm{Diff}^\Omega(F)$ acts freely on $\mathrm{Emb}^\Omega(F; [0, t] \times \mathbb{R}^\infty)$ by

$$(\phi.h)(x) = h(\phi^{-1}(x)).$$

Define the space of morphisms in $\mathcal{Y}$ to be the space of orbits

$$\text{(2.1)} \qquad \mathrm{morph}\, \mathcal{Y} = \mathrm{ob}\, \mathcal{Y} \sqcup \coprod_{F, t>0} \mathrm{Emb}^\Omega(F; [0, t] \times \mathbb{R}^\infty)/\mathrm{Diff}^\Omega(F),$$

where the disjoint union is taken over $t > 0$ and cobordisms F, one fixed representative for each diffeomorphism type of cobordisms from S_n to S_m. The diffeomorphism types can be described as follows. Write F as the disjoint union of its connected components,

$$\text{(2.2)} \qquad F = F_{g_1, n_1+m_1} \sqcup \cdots \sqcup F_{g_k, n_k+m_k}.$$

Here $k \geq 1$, $g_i, n_i, m_i \geq 0$, $\Sigma_i n_i = n$, and $\Sigma_i m_i = m$. The diffeomorphism type of F as a cobordism is determined by the unordered k-tuple of triples (g_i, n_i, m_i). The topology on the morphism space is compatible with the usual topology on $\mathbb{R}^+$ and the topology of the embedding spaces. The elements of ob $\mathcal{Y}$ represent the identity morphisms. We may think of a morphism $[h] = \mathrm{im}(h)$ in $\mathcal{Y}$ as an embedded, oriented, unparametrized cobordism in $[0, t] \times \mathbb{R}^\infty$.

Structure maps. The source and target functions are induced by the restriction functions ∂_- and ∂_+. Given two embedded surfaces $[h_1], [h_2]$ with $[\partial_+ h_1] = [\partial_- h_2] = [\beta : S_m \to \mathbb{R}^\infty]$, their composition is

(2.3) $$[h_2] \circ [h_1] = \mathrm{im}(h_1) \sqcup_{\mathrm{im}(\beta)} \mathrm{im}(h_2) \subset [0, t_1 + t_2] \times \mathbb{R}^\infty.$$

To see that this is well-defined, pick representatives $h_i : F_i \to [0, t_i] \times \mathbb{R}^\infty$ and let $\alpha = (\partial_- h_2)^{-1} \circ (\partial_+ h_1) : S_m \to S_m$. Glue F_1 and F_2 along their common boundary S_m using α to form a surface $F_1 \sqcup_\alpha F_2$. For the representative F of $F_1 \sqcup_\alpha F_2$ in (2.1) pick a diffeomorphism of cobordisms $\phi : F \to F_1 \sqcup_\alpha F_2$. Define

$$[h_2] \circ [h_1] = [(h_1 \sqcup h_2) \circ \phi],$$

with $h_1 \sqcup h_2 : F_1 \sqcup_\alpha F_2 \to [0, t_1 + t_2] \times \mathbb{R}^\infty$. This definition is independent of the choices of h_1, h_2 and ϕ. One checks that composition is continuous.

We next determine the homotopy type of the morphism spaces $\mathcal{Y}([\alpha], [\beta])$ for any embeddings $\alpha : S_n \hookrightarrow \mathbb{R}^\infty$ and $\beta : S_m \hookrightarrow \mathbb{R}^\infty$. For F connected the restriction maps ∂_- and ∂_+ define a fibration

$$\mathrm{Diff}(F; \partial) \longrightarrow \mathrm{Diff}^\Omega(F) \xrightarrow{(\partial_-, \partial_+)} \mathrm{Diff}(S_n) \times \mathrm{Diff}(S_m).$$

When F is not connected , the map (∂_-, ∂_+) may fail to be surjective on components. Let $\Sigma_F \subset \Sigma_n \times \Sigma_m$ denote the subgroup that is the image under the restriction map (∂_-, ∂_+) of the connected components in $\mathrm{Diff}^\Omega(F)$. For F as in (2.2), when all triples (g_i, n_i, m_i) are distinct, Σ_F is the group $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k} \times \Sigma_{m_1} \times \cdots \times \Sigma_{m_k}$. If not all triples (g_i, n_i, m_i) are distinct, then $\mathrm{Diff}^\Omega(F)$ contains the diffeomorphisms that permute any diffeomorphic factors of F. These diffeomorphisms correspond to certain block permutations in Σ_F. In general there is a fibration

(2.4) $$\mathrm{Diff}(F; \partial) \longrightarrow \mathrm{Diff}^\Omega(F) \xrightarrow{(\partial_-, \partial_+)} \Sigma_F \wr \mathrm{Diff}(S^1).$$

Theorem 2.1. *There are homotopy equivalences*

$$\mathcal{Y}([\alpha], [\beta]) \simeq \coprod_F B\mathit{Diff}(F; \partial) \times (\Sigma_n \times \Sigma_m)/\Sigma_F \quad \text{when } [\alpha] \neq [\beta],$$

$$\mathcal{Y}([\alpha], [\beta]) \simeq id_{[\alpha]} \sqcup \coprod_F B\mathit{Diff}(F; \partial) \times (\Sigma_n \times \Sigma_m)/\Sigma_F \quad \text{when } [\alpha] = [\beta],$$

where the disjoint union is taken over all F as in (2.1) and $id_{[\alpha]}$ is the identity morphism of $[\alpha] \in ob\ \mathcal{Y}$.

Proof. Let

$$\mathrm{Emb}^\Omega_{\alpha,\beta}(F; [0, t] \times \mathbb{R}^\infty), \quad \mathrm{Emb}^\Omega_{[\alpha],[\beta]}(F; [0, t] \times \mathbb{R}^\infty)$$

denote the subspace of embeddings h such that

$$\partial_- h = \alpha \text{ and } \partial_+ h = \beta, \quad \text{respectively,} \quad [\partial_- h] = [\alpha] \text{ and } [\partial_+ h] = [\beta].$$

Restriction to the boundary defines a fibration

$$(2.5)\qquad \mathrm{Emb}^\Omega(F;[0,t]\times\mathbb{R}^\infty) \xrightarrow{(\partial_-,\partial_+)} \mathrm{Emb}\big(S_n,\mathbb{R}^\infty\big)\times\mathrm{Emb}\big(S_m,\mathbb{R}^\infty\big)$$

whose fibre over (α,β) is $\mathrm{Emb}^\Omega_{\alpha,\beta}(F;[0,t]\times\mathbb{R}^\infty)$. It restricts to a fibration

$$\mathrm{Emb}^\Omega_{[\alpha],[\beta]}(F;[0,t]\times\mathbb{R}^\infty) \xrightarrow{(\partial_-,\partial_+)} (\mathrm{Diff}(S_n)\times\mathrm{Diff}(S_m))(\alpha,\beta)$$

with base the free orbit of (α,β). The fibration (2.4) of groups acts freely on this restricted fibration. On orbits one obtains a fibration

$$(2.6)\qquad \mathrm{Emb}^\Omega_{[\alpha],[\beta]}(F;[0,t]\times\mathbb{R}^\infty)/\mathrm{Diff}^\Omega(F) \longrightarrow (\Sigma_n\times\Sigma_m)/\Sigma_F$$

with fibre $\mathrm{Emb}^\Omega_{\alpha,\beta}(F;[0,t]\times\mathbb{R}^\infty)/\mathrm{Diff}(F;\partial)$. Any connected component of the space of embeddings $\mathrm{Emb}(M,\partial M;N,\partial N)$ of a manifold M into an m-connected manifold N is $(n-2-2m)$-connected where $m=\dim M$ and $n=\dim N$. This can be derived from Whitney's embedding theorems, Theorem 5 and 6 in [W]. Thus $\mathrm{Emb}^\Omega_{\alpha,\beta}(F;[0,t]\times\mathbb{R}^\infty)$ is contractible, and hence the fibre of (2.6) is homotopic to

$$B\mathrm{Diff}(F;\partial).$$

The total space of (2.6) injects homotopy equivalently into a connected component of $\mathcal{Y}([\alpha],[\beta])$. This proves the theorem. □

2.2. *Deloop of $\mathbb{Z}\times B\Gamma^+_\infty$ and Γ-space structure*

Consider now the subcategory

$$\mathcal{Y}_b\subset\mathcal{Y}$$

of $\mathcal{Y}$ with the same object space, and morphism space just as in (2.1) except that the disjoint union is taken over cobordisms F of type (2.2) such that each m_i is positive, i.e. each connected component of F has at least one outgoing boundary circle.

Given a pointed space X, let ΩX denote its based loop space, and given a category $\mathcal{C}$, let $|\mathcal{C}|$ denote the realization of its simplical nerve $N.\mathcal{C}$. Pick any object to represent the base point in $|\mathcal{C}|$.

Theorem 2.2. *There is a homotopy equivalence $\Omega|\mathcal{Y}_b|\simeq\mathbb{Z}\times B\Gamma^+_\infty$.*

Proof. The argument in [T2] can be adopted to the present situation and we refer to that paper for more details. Fix $\beta_0:S_1\hookrightarrow\mathbb{R}^2\subset\mathbb{R}^\infty$ and let $\alpha:S_n\hookrightarrow\mathbb{R}^\infty$ represent any other object in $\mathcal{Y}$. In $\mathcal{Y}_b$ any cobordism from $[\alpha]$ to $[\beta_0]$ has to be connected, and by Theorem 2.1

$$\mathcal{Y}_b([\alpha],[\beta_0])\simeq\coprod_{g\geq 0}B\Gamma_{g,n+1},$$

with the identity morphism added when $[\alpha] = [\beta_0]$. Fix a morphism $[h_0] \in \mathcal{Y}_b([\beta_0], [\beta_0])$ of type $F_{1,1+1}$ and form the telescope

(2.7)

$$\mathcal{X}_\infty([\alpha]) = \mathrm{Tel}(\mathcal{Y}_b([\alpha], [\beta_0]) \xrightarrow{[h_0]} \mathcal{Y}_b([\alpha], [\beta_0]) \xrightarrow{[h_0]} \dots) \simeq \mathbb{Z} \times B\Gamma_{\infty,n}.$$

Let $E_{\mathcal{Y}_b}\mathcal{X}_\infty$ be the homotopy colimit of the contravariant functor $\mathcal{X}_\infty$ from $\mathcal{Y}_b$ to topological spaces. It can be identified with the telescope (under $[h_0]$) of the classifying space of the category of objects over $[\beta_0]$ in $\mathcal{Y}_b$. Hence it is contractible. By the homology stability theorem [H2], [I], cf. (1.3), the homology of $B\Gamma_{g,n+1}$ is independent of g and n in dimensions less that $g/2$. Thus the natural projection

$$E_{\mathcal{Y}_b}\mathcal{X}_\infty \longrightarrow B\mathcal{Y}_b = |\mathcal{Y}_b|$$

is a homology fibration. By the group completion theorem for categories (see [T2] and [McS]) this gives a homotopy equivalence

$$\mathcal{X}_\infty([\alpha])^+ \simeq \mathbb{Z} \times B\Gamma_\infty^+ \longrightarrow \Omega|\mathcal{Y}_b|$$

for any $[\alpha] \in \mathcal{Y}_b$. □

Remark. The above proof does not extend as to give the homotopy type of $\Omega|\mathcal{Y}|$. However, from [T1] we know that $\Omega|\mathcal{Y}_b|$ and $\Omega|\mathcal{Y}|$ have the same group of connected components, and one might expect that they are homotopy equivalent.

Let Γ^{op} be the category of finite based sets

$$\mathbf{n} = \{0, 1, \dots, n\}, \qquad n \geq 0,$$

and based maps. A Γ-space in the sense of Segal [S1] is a functor $\mathbf{F}$ from Γ^{op} to the category of simplicial spaces such that the map

$$p_1 \times \cdots \times p_n : \mathbf{F}(\mathbf{n}) \longrightarrow \mathbf{F}(\mathbf{1}) \times \cdots \times \mathbf{F}(\mathbf{1}), \qquad n-\text{factors},$$

is a homotopy equivalence for each $n \geq 0$; here the map $p_i : \mathbf{n} \to \mathbf{1}$ sends i to 1 and $j \neq i$ to the base point 0. The main theorem of [S1] is that if $\pi_0\mathbf{F}(\mathbf{1})$ is a group then $\mathbf{F}(\mathbf{1})$ is an infinite loop space.

We will now define a Γ-structure corresponding to disjoint union of cobordisms on the simplicial nerve $N.\mathcal{Y}$ of $\mathcal{Y}$ which restricts to a Γ-structure on the nerve $N.\mathcal{Y}_b$ of $\mathcal{Y}_b$. Let $\mathbf{Y}$ be a functor from Γ^{op} to simplical spaces such that the q simplices in $\mathbf{Y}(\mathbf{n})$ are the pairs

$$([\bar{h}], \lambda) \quad \text{with } [\bar{h}] \in N_q\mathcal{Y} \text{ and } \lambda : \pi_0\mathrm{im}(\bar{h}) \to \{1, \dots, n\}$$

where $[\bar{h}] = ([h_1], \dots, [h_q])$ is a q-tuple of composable morphisms, and $\mathrm{im}(\bar{h}) := [h_q] \circ \cdots \circ [h_1]$ is the composed embedded surface, cf. (2.3). In particular, $\mathbf{Y}(\mathbf{0}) = *$, represented by the empty manifold $\emptyset = S_0$, and $\mathbf{Y}(\mathbf{1}) = N.\mathcal{Y}$. For $s : \mathbf{n} \to \mathbf{m}$ in Γ^{op} define

$$\mathbf{Y}(s) : \mathbf{Y}(\mathbf{n}) \longrightarrow \mathbf{Y}(\mathbf{m}) \quad \text{via} \quad ([\bar{h}], \lambda) \mapsto ([\bar{h}], s \circ \lambda).$$

Theorem 2.3. **Y** *is a* Γ*-space, and hence* $\mathbb{Z} \times B\Gamma^{+}_{\infty}$ *is an infinite loop space.*

Proof. The projection p_i maps $([\bar{h}], f) \in \mathbf{Y}(\mathbf{n})_q$ to the union of the components in $f^{-1}(i)$. Hence, $\mathbf{Y}(\mathbf{n})_q$ is mapped injectively onto $\mathbf{Y}(\mathbf{1})^n_q \setminus \triangle$ where $\triangle$ is the fat diagonal of elements $([\bar{h}^1], \dots, [\bar{h}^n])$ in $\mathbf{Y}(\mathbf{1})^n_q$ such that $\mathrm{im}(\bar{h}^i) \cap \mathrm{im}(\bar{h}^j) \neq \emptyset$ for some $i \neq j$. Observe that for any compact set $K \subset [0, t] \times \mathbb{R}^{\infty}$, the inclusion

$$\mathrm{Emb}^{\Omega}(F; [0, t] \times \mathbb{R}^{\infty} \setminus K) \hookrightarrow \mathrm{Emb}^{\Omega}(F; [0, t] \times \mathbb{R}^{\infty})$$

is a homotopy equivalence of contractible spaces by Whitney's embedding theorems [loc.cit.]. Clearly, the inclusion is $\mathrm{Diff}^{\Omega}(F)$-equivariant and hence induces a homotopy equivalence on homotopy orbits. But the action of $\mathrm{Diff}^{\Omega}(F)$ is free on both spaces so that it induces also a homotopy equivalence on orbits. An inductive argument shows that $\mathbf{Y}(\mathbf{1})^n \setminus \triangle \hookrightarrow \mathbf{Y}(\mathbf{1})^n$ is a homotopy equivalence. Thus **Y** defines a Γ-space structure on $N.\mathcal{Y}$ which clearly restricts to a Γ-space structure on $N\mathcal{Y}_b$. Note that $|\mathcal{Y}|$ and $|\mathcal{Y}_b|$ are connected and hence by [S1] are infinite loop spaces. But by Theorem 2.2 $\Omega|\mathcal{Y}_b| \simeq \mathbb{Z} \times B\Gamma^{+}_{\infty}$ and thus $\mathbb{Z} \times B\Gamma^{+}_{\infty}$ is an infinite loop space. □

2.3. Deloop of α_{∞}

We will first give a convenient model for the deloop of $\Omega^{\infty}\mathbb{C}P^{\infty}_{-1}$. Let $\mathrm{Gr}(2, 2l+2)$ denote the Grassmannian manifold of oriented 2-dimensional subspaces P in $\mathbb{R}^{2l+2}$. The orthogonal complement $-L_l$ of the canonical plane bundle over $\mathrm{Gr}(2, 2l+2)$ has fibre

$$\{v \in \mathbb{R}^{2l+2} | P \perp v\}$$

at P. Let $\mathrm{Th}(-L_l)$ be its Thom space, i.e. the fibrewise campactification of $-L_l$ modulo the section at infinity. Define a simplicial space $Z^l.$ by setting $Z^l_0 = \Omega^{2l+1}\mathrm{Th}(-L_l)$ and for $q \geq 1$, Z^l_q to be the space of $(q-1)$-times broken paths in $\Omega^{2l+1}\mathrm{Th}(-L_l)$. To be more precise and fix the notation

$$Z^l_q := \coprod_{u \in U_q} \mathrm{Map}([0, u(q)]; \Omega^{2l+1}\mathrm{Th}(-L_l)),$$

where the disjoint union is taken over the space U_q of monotone, based maps $u : \{0, 1, \dots, q\} \to \mathbb{R}_{\geq 0}$. The topology on Z^l_q is compatible with the usual topology of paths in $\Omega^{2l+1}\mathrm{Th}(-L_l)$ and the topology on U_q. The simplicial maps are induced by the based monotonic maps $\{0, \dots, q\} \to \{0, \dots, p\}$ in the usual way. There are canonical maps $Z^l. \to Z^{l+1}.$ of simplicial spaces induced by adding the trivial line bundle to $-L_l$. Define the simplicial space $Z.$ by

$$Z. := \mathrm{colim}_{\,l \to \infty} Z^l.$$

Lemma 2.4. *There is a homotopy equivalence* $|Z.| \simeq \Omega^{\infty-1}\mathbb{C}P^{\infty}_{-1}$.

Proof. Consider $Z^l_0 = \Omega^{2l+1}\mathrm{Th}(-L_l)$ as the constant simplicial space and let $I : Z^l_0 \to Z^l.$ be the simplicial map that identifies Z^l_0 at the q-th level with the constant maps in $\mathrm{Map}([0, u(q)]; \Omega^{2l-1}\mathrm{Th}(-L_l)) \subset Z^l_q$ where u is defined by $u(i) = i$. Each U_q is contractible and each $\mathrm{Map}([0, u(q)], \Omega^{2l+1}\mathrm{Th}(-L_l))$ is homotopic to $\Omega^{2l+1}\mathrm{Th}(-L_l)$. Thus the simplical map I defines is a homotopy equivalence in each simplicial dimension and hence

$$|Z^l.| \simeq \Omega^{2l+1}\mathrm{Th}(-L_l).$$

Recall that by definition $\Omega^{\infty}(\mathbb{C}P^{\infty}_{-1}) = \mathrm{colim}\ \Omega^{2l-2}\mathrm{Th}(-L_l)$. Hence,

$$|Z.| = \Omega^{\infty-1}\big(\mathbb{C}P^{\infty}_{-1}\big).$$

□

Define a simplicial Γ-space $\mathbf{Z}.$ by setting $\mathbf{Z}(\mathbf{n})_q$ to be the pairs

$$(f, \sigma) \quad \text{with} \quad f \in Z_q \quad \text{and} \quad \sigma : \mathrm{supp}(f) \to \{1, \ldots, n\},$$

where f is considered as a function $[0, u(q)]_+ \wedge S^{2l+1} \to \mathrm{Th}(-L^l)$ and $\mathrm{supp}(f)$ is the complement of $f^{-1}(*)$, the inverse image of the point at infinity in $\mathrm{Th}(-L_l)$. For $s : \mathbf{n} \to \mathbf{m}$ in Γ^{op} define $\mathbf{Z}(s)$ via $(f, \sigma) \mapsto (f, s \circ \sigma)$. By a similar argument as for $\mathbf{Y}$, one sees that this defines an infinite loop space structure on $|Z.|$ which is compatible with the usual infinite loop space structure on $\Omega^{\infty-1}\mathbb{C}P^{\infty}_{-1}$.

In order to define the Thom collapse map, we need to consider embedded surfaces with a tubular neighborhood. In analogy with Z_q, we may think of a q-simplex $[\bar{h}]$ in $N_q\mathcal{Y}$ as a q-times broken cobordism in $[0, u(q)] \times \mathbb{R}^{2l+1}$, for some l, with breaking points $u(i) = t_1 + \cdots + t_i$. Let Y^l_q be the space of pairs

$$\zeta = ([\bar{h}], \mathcal{O}) \tag{2.8}$$

where $[\bar{h}]$ is as above and $\mathcal{O}$ is a tubular neighborhood of $\mathrm{im}(\bar{h}) = [h_q] \circ \cdots \circ [h_1]$ in $[0, u(q)] \times \mathbb{R}^{2l+1}$. For a given $[\bar{h}]$, the space of tubular neighborhoods is contractible. The canonical projections $Y^l. \to N.\mathcal{Y}$ define thus a homotopy equivalence

$$|Y.| \xrightarrow{\simeq} |\mathcal{Y}|, \quad \text{with} \quad Y. := \mathrm{hocolim}_{l\to\infty} Y^l. \tag{2.9}$$

The Γ-space structure on $N.\mathcal{Y}$ extends to a Γ-space structure on $Y.$ in an obvious way. In particular (2.9) is a homotopy equivalence of infinite loop spaces which restricts to $|Y_b.| \simeq |\mathcal{Y}_b|$ where $Y_b.$ is the subspace of $Y.$ corresponding to $\mathcal{Y}_b$.

We can now define a simplicial map

$$\theta : Y. \to Z.$$

using Thom collapse and the classifying map of the vertical tangent bundle for the universal surface bundle. Let

$$\zeta = ([\bar{h}], \mathcal{O}) \in Y_q^l$$

be as in (2.8). Identify $\mathcal{O}$ via radial expansion with the normal bundle $N\text{im}(\bar{h})$ of the embedded surface $\text{im}(\bar{h})$. Under this identification, let the point $(s,t) \in \mathcal{O} \subset [0, u(q)] \times \mathbb{R}^{2l+1}$ correspond to (x, v) with $x \in \text{im}(\bar{h})$ and $v \in N_x\text{im}(\bar{h})$. Define

$$\theta_\zeta \in \text{Map}\big([0, u(q)]_+ \wedge S^{2l+1}, \text{Th}(-L_l)\big)$$

by

$$\theta_\zeta(s,t) = \begin{cases} * & \text{if} \quad (s,t) \notin \mathcal{O} \\ (T_x\text{im}(\bar{h}), v) & \text{if} \quad (s,t) \in \mathcal{O} \end{cases}$$

where $T_x\text{im}(\bar{h})$ is the tangent plane at x. The point corresponding to the empty set (with the empty tubular neighborhood) is mapped by definition to the constant map at infinity $*$.

Clearly θ is compatible with the Γ-structures on $Y.$ and $Z.$. Restricting θ to $Y_b.$ and taking loop spaces, we have proved

Theorem 2.5. $\alpha_\infty := \Omega(\theta|_{Y_b.}) : \mathbb{Z} \times B\Gamma^+_\infty \to \Omega^\infty \mathbb{C}P^\infty_{-1}$ *is a map of infinite loop spaces.* □

2.4. Induced map of components and relation to the transfer map

The categories $\mathcal{Y}$ and $\mathcal{Y}_b$ have many objects and hence $|\mathcal{Y}|$ and $|\mathcal{Y}_b|$ have no canonical base point. The point representing the empty 1-manifold $\emptyset = S_0$ is however the implicit choice in Theorem 2.5 as it is the only vertex mapped by θ to the base point in $\Omega^{\infty-1}(\mathbb{C}P^\infty_{-1})$. The empty set $\emptyset$ is also the unit for the Γ-space structure on $N.\mathcal{Y}$.

From the proof of Theorem 2.2 it makes most sense to identify the components in $\Omega|\mathcal{Y}_b| \simeq \mathbb{Z} \times B\Gamma^+_\infty$ with the integers in such a way that the translation by $[h_0]$ in (2.7) represents component shift by $+1$, i.e. components are linked to the genus g. More precisely, let F be a cobordism from S_n to S_m, and let $\alpha : S_n \to \mathbb{R}^\infty$ and $\beta : S_m \to \mathbb{R}^\infty$ represent two objects in $\mathcal{Y}$. By (2.6) there is a map

$$B\text{Diff}(F; \partial) \simeq \text{Emb}^\Omega_{\alpha,\beta}(F; [0,1] \times \mathbb{R}^\infty)/\text{Diff}(F; \partial) \hookrightarrow \mathcal{Y}([\alpha], [\beta]),$$

and hence a map

$$[0, 1] \times B\mathrm{Diff}(F; \partial) \longrightarrow |\mathcal{Y}|.$$

Taking adjoints yields a map

$$B\mathrm{Diff}(F; \partial) \longrightarrow \Omega^{[\alpha],[\beta]}|\mathcal{Y}|$$

where $\Omega^{x,y}$ denotes the space of paths from x to y. Combined with any choice of morphisms

$$[h : F' \to [0, 1] \times \mathbb{R}^\infty] \in \mathcal{Y}(\emptyset, [\alpha])$$

and

$$[h' : F'' \to [0, 1] \times \mathbb{R}^\infty] \in \mathcal{Y}(\emptyset, [\beta])$$

this defines a map from $B\mathrm{Diff}(F; \partial)$ to $\Omega^{\emptyset,\emptyset}|\mathcal{Y}| = \Omega|\mathcal{Y}|$. The component of the image depends on the choice of $[h]$ and $[h']$. By [T1; Theorem 7] this component is given by

$$\frac{1}{2}(\chi(F'') - \chi(F) - \chi(F')) \quad \in \mathbb{Z} = \pi_0\, \Omega|\mathcal{Y}_b| = \pi_0\, \Omega|\mathcal{Y}| \tag{2.10}$$

where χ denotes the Euler characteristic.

We now define two *group completion maps*. For $F = F_{g,1+1} : S_1 \to S_1$, let $F' = F'' : S_0 \to S_1$ be the disk, and $[\alpha] = [\beta]$ be represented by the unit circle in $\mathbb{R}^2 \subset \mathbb{R}^\infty$. This yields a group completion map into the g-th component

$$\gamma_b : B\mathrm{Diff}(F_{g,1+1}; \partial) \longrightarrow \Omega|\mathcal{Y}_b| \simeq \mathbb{Z} \times B\Gamma_\infty^+.$$

When $F = F_g$ is the closed surface we need to consider the full category $\mathcal{Y}$ in order to define the group completion map. Setting $F' = F'' = Id_\emptyset$ this gives a map into the $(g-1)$-th component:

$$\gamma : B\mathrm{Diff}(F_g) \longrightarrow \Omega|\mathcal{Y}|.$$

γ_b and γ are related by the homotopy commutative diagram

$$\begin{array}{ccc} B\mathrm{Diff}(F_{g,1+1}; \partial) & \xrightarrow{\mathrm{incl}} & B\mathrm{Diff}(F_g) \\ {\scriptstyle \gamma_b}\downarrow & & {\scriptstyle \gamma}\downarrow \\ \Omega|\mathcal{Y}_b| & \xrightarrow{-[1]} & \Omega|\mathcal{Y}| \end{array} \tag{2.11}$$

where incl denotes the inclusion map induced by gluing two disks to the boundary components of $F_{g,1+1}$ and $\pm[k]$ denotes component shift by $\pm k$.

Define

$$\alpha_{F_g} = \Omega(\theta) \circ \gamma : B\mathrm{Diff}(F_g) \longrightarrow \Omega^\infty \mathbb{C}P^\infty_{-1} \tag{2.12}$$

and compare it with the Becker-Gottlieb transfer map of the universal surface bundle, cf. (1.2) and (1.6). Taking the quotient space $\mathrm{Emb}(F_g,[0,1]\times\mathbb{R}^\infty)/\mathrm{Diff}(F_g)\subset Y_1$ as our model for $B\mathrm{Diff}(F_g)$, the transfer

$$\begin{aligned}\mathrm{trf}(\pi_{F_g}) : B\mathrm{Diff}(F_g) \to \mathrm{Th}(N^vE(F_g)) &\to \mathrm{Th}(T^vE(F_g)\oplus N^vE(F_g))\\ &\simeq Q(E(F_g)_+)\end{aligned}$$

can be defined by assigning to $\zeta=([h],\mathcal{O})\in Y_1$ the collapse map $\tilde{c}_\zeta$ which is defined for $(s,t)\in[0,1]\times\mathbb{R}^\infty$ by

$$\tilde{c}_\zeta(s,t)=\begin{cases} * & \text{if} \quad (s,t)\notin\mathcal{O}\\ (x,v) & \text{if} \quad (s,t)\in\mathcal{O};\end{cases}$$

here, as in the previous section, the neighborhood $\mathcal{O}$ of $\mathrm{im}(h)$ is identified with the normal bundle $N\mathrm{im}(h)$ such that (s,t) corresponds to (x,v) with $x\in\mathrm{im}(h)$ and $v\in N_v\mathrm{im}(h)$. From the definition of θ it follows that the diagram

$$\begin{array}{ccccc} B\mathrm{Diff}(F_g) & \xrightarrow{\mathrm{trf}(\pi_{F_g})} & Q(E(F_g)_+) & \xrightarrow{Q(T^v_+)} & Q(BS^1_+)\\ {\scriptstyle\gamma}\downarrow & & & & \|\\ \Omega|\mathcal{Y}|\simeq\Omega|Y.| & \xrightarrow{\Omega(\theta)} & \Omega|Z.|\simeq\Omega^\infty\mathbb{C}P^\infty_{-1} & \xrightarrow{\omega_\infty} & Q(BS^1_+)\end{array}$$

is homotopy commutative; here T^v denotes the classifying map of the vertical tangent bundle $T^vE(F_g)$. Hence,

$$\tau_{F_g}=Q(T^v_+)\circ\mathrm{trf}(\pi_{F_g})\simeq\omega_\infty\circ\alpha_{F_g}. \tag{2.13}$$

Lemma 2.6. *The map $\pi_0(\Omega(\theta))=\pi_0(\alpha_\infty):\mathbb{Z}\to\mathbb{Z}$ is multiplication by -1.*

Proof. The transfer map $\mathrm{trf}(\pi):B_+\to Q(E_+)$ of a fibration $F\to E\xrightarrow{\pi}B$ (with B and F homotopy equivalent to finite CW-complexes) sends the base space to the connected component of the Euler characteristic of its fibre, cf. [BG]. As by definition $Q(T^v_+)$ preserves components, (2.10) and (2.13) imply that

$$\pi_0(\omega_\infty\circ\Omega(\theta))=\pi_0(\omega_\infty\circ\alpha_\infty)=\text{multiplication by }-2. \tag{2.14}$$

In degree 0 the homotopy exact sequence of the fibration (1.7),

$$0\longrightarrow\pi_0(\Omega^\infty\mathbb{C}P^\infty_{-1})\xrightarrow{\omega_\infty}\pi_0(Q(BS^1_+))\xrightarrow{\partial}\pi_1(QS^0)\longrightarrow 0,$$

is

$$0\longrightarrow\mathbb{Z}\longrightarrow\mathbb{Z}\longrightarrow\mathbb{Z}/2\longrightarrow 0.$$

Hence $\pi_0(\omega_\infty)$ is multiplication by 2 from $\mathbb{Z}$ to $\mathbb{Z}$, and $\pi_0(\alpha_\infty)=\pi_0(\Omega(\theta))$ is multiplication by -1. □

On composition with $\Omega(\theta)$ and in turn with ω_∞, (2.11) yields homotopies

$$\begin{aligned} \alpha_{F_g} \circ \mathrm{incl} &\simeq +[1] \circ \alpha_\infty \circ \gamma_b : B\mathrm{Diff}(F_{g,1+1}; \partial) \longrightarrow \Omega^\infty \mathbb{C}P^\infty_{-1}, \\ \tau_{F_g} \circ \mathrm{incl} &\simeq +[2] \circ \tau_\infty \circ \gamma_b : B\mathrm{Diff}(F_{g,1+1}; \partial) \longrightarrow Q\big(BS^1_+\big). \end{aligned} \tag{2.15}$$

3. Transfers and diffeomorphisms of surfaces

This section constructs surfaces equipped with an action of a cyclic group C_q and exhibits C_q-invariant vector fields on them. We then apply the "parametrised Poincaré-Hopf" theorem from [BM] to get information about the transfer of the universal smooth surface bundle (1.2).

3.1. Branched covers

Let Σ be a (fixed) connected surface. We consider divisors $D = \Sigma_{i=0}^k n_i p_i \in C_0(\Sigma; \mathbb{Z}/q)$ with

$$(n_i, q) = 1 \ \text{ and } \ \Sigma_{i=0}^k n_i \equiv 0 \ (\mathrm{mod} q). \tag{3.1}$$

Given D, we construct an associated connected surface F with a smooth C_q-action, and

$$F^{C_q} = \{p_0, \dots, p_k\}, \quad F/C_q = \Sigma.$$

Let $\mathbb{C}(n)$ denote the complex plane with C_q-action $t \cdot z = e^{2\pi i n/q} \cdot z$ where $t \in C_q$ is a generator. The tangent representation at p_i of the surface F will be

$$T_{p_i} F = \mathbb{C}(m_i), \quad m_i n_i \equiv 1 \ (\mathrm{mod} q). \tag{3.2}$$

To construct F, consider the complement Σ^* of a small open tube $N\{p_0, \dots, p_k\}$ of the branch points. We have the Poincaré duality diagram

$$\begin{array}{ccc} H^1(\Sigma^*; \mathbb{Z}/q) & \xrightarrow{\delta^*} & H^2(\Sigma, \Sigma^*; \mathbb{Z}/q) \\ \simeq\big\downarrow & & \simeq\big\downarrow \\ H_1(\Sigma, \{p_0, \dots, p_k\}; \mathbb{Z}/q) & \xrightarrow{\partial_*} & H_0(\{p_0, \dots, p_k\}; \mathbb{Z}/q), \end{array}$$

and note by excision that

$$H^2(\Sigma, \Sigma^*; \mathbb{Z}/q) \simeq \bigoplus_{i=0}^k H^2\big(D^2_{p_i}, S^1_{p_i}; \mathbb{Z}/q\big).$$

The second condition in (3.1) implies a class $\kappa_D \in H^1(\Sigma^*; \mathbb{Z}/q)$ with $\partial_*(\kappa_D \cap [\Sigma]) = D$; the class κ_D is not unique except when Σ is the 2-sphere. We view κ_D as a map from Σ^* to BC_q and let $F^* \to \Sigma^*$ be the induced principal C_q-cover.

Let $S^1(m)$ (resp. $D^2(m)$) be the unit C_q-sphere (resp. -disk) of $\mathbb{C}(m)$, and let $\Delta_n : S^1 \to S^1$ be the n-th power map $\Delta_n(z) = z^n$. The restriction of the C_q-cover $F^* \to \Sigma^*$ to the i-th boundary component $S^1_{p_i}$ is by construction the pull-back

$$\begin{array}{ccc} \partial_i F^* & \longrightarrow & S^1(1) \\ \big\downarrow & & \big\downarrow{\scriptstyle \Delta_q} \\ S^1_{p_i} & \xrightarrow{\Delta_{n_i}} & S^1, \end{array}$$

so $\partial_i F^* = \{(w, z) | w^{n_i} = z^q\}$. This is a circle by associating to $u \in S^1$ the pair (u^q, u^{n_i}). Thus $\partial_i F^* = S^1(m_i)$ as a C_q-space, where $m_i n_i \equiv 1 \pmod{q}$.

Definition 3.1. *The C_q-branched cover of Σ associated with the divisor D is the surface*

$$F = F^* \sqcup_\partial (D(m_0) \sqcup \ldots \sqcup D(m_k)).$$

In preparation for Theorem 3.5 below we note the following.

Lemma 3.2. *There exists a non-degenerate C_q-invariant vector field X on F whose singular set $S(X)$ contains the branch points $\{p_0, \ldots, p_k\}$ with local indices*

$$\operatorname{ind}_{p_i}(X) = +1.$$

Proof. We choose a Morse function $f : \Sigma \to \mathbb{R}$ such that $\{p_0, \ldots, p_k\}$ are local maxima or local minima. Let $\bar{X}$ be its gradient vector field. Its singular set includes the branch points, so it lifts to an equivariant vector field X on F.

The local index of $\bar{X}$ at p_i is $+1$, since the Morse index at p_i is ± 2, and thus $\operatorname{ind}_{p_i}(X)$ is also $+1$. (If the Morse index for f at p_i had been ± 1, then $\operatorname{ind}_{p_i}(\bar{X}) = -1$ and $\operatorname{ind}_{p_i}(X) = 1 - 2q$ so X would have been degenerate at p_i). For $p \in S(X) \setminus \{p_0, \ldots, p_k\}$, $\operatorname{ind}_p(X) = \operatorname{ind}_{\pi(p)}(\bar{X})$. This completes the proof. □

3.2. Transfers and vector fields

The transfer map of [BG], cf. (1.6), associates to a fibration $F \to E \xrightarrow{\pi} B$, with B and F homotopy equivalent to finite CW-complexes, a unique homotopy class

$$\operatorname{trf}(\pi) : B_+ \longrightarrow Q(E_+).$$

We shall need the following property of the transfer valid when $E \xrightarrow{\pi} B$ has compact Lie structure group G and F admits a non-degenerate G-invariant vector field X. Its singular set $S(X)$ is a finite G-set, so there is a finite covering space contained in $E \to B$:

$$\begin{array}{ccc} P \times_G S(X) & \xrightarrow{\text{incl}} & P \times_G F = E \\ {\scriptstyle \pi_S}\downarrow & & {\scriptstyle \pi}\downarrow \\ B & = & B. \end{array}$$

Theorem 2.10 from [BM] asserts a relationship between the Becker-Gottlieb transfers $\mathrm{trf}(\pi_S)$ and $\mathrm{trf}(\pi)$:

Theorem 3.3 [BM]. *There is a homotopy commutative diagram*

$$\begin{array}{ccc} B & \xrightarrow{\mathrm{trf}(\pi)} & Q(P \times_G F_+) \\ {\scriptstyle \mathrm{trf}(\pi_S)}\downarrow & & \uparrow{\scriptstyle Q(\mathrm{incl}_+)} \\ Q(P \times_G S(X)_+) & \xrightarrow{\mathrm{IND}(X)} & Q(P \times_G S(X)_+). \end{array}$$

□

We recall the definition of $\mathrm{IND}(X)$. Choose a G-invariant metric on F, a G-embedding of F into a representation space V, and a complement η to the vector bundle $P \times_G V$. This is possible since B is compact. For $\sigma \in S(X)$, the differential dX_σ may be considered as an automorphism of the tangent space $T_\sigma F$, and gives rise to a G-bundle automorphism dX of $\tau_F|_{S(X)} = S(X) \times V$. We add the identity on $\nu_F|_{S(X)}$ and apply $P \times_G (-)$ to get a bundle automorphism

$$P \times_G dX : P \times_G (S(X) \times V) \longrightarrow P \times_G (S(X) \times V)$$

over $P \times_G S(X)$. Let $\dot{\eta}$ denote the fibrewise one point compactification of η. Then

$$\left(P \times_G S(X) \times S^V\right) \wedge_{P \times_G S(X)} \pi_S^*(\dot{\eta}) \simeq (P \times_G S(X)_+) \wedge S^m$$

where on the left we take the fibrewise smash product. The fibrewise one point compactification of $P \times_G dX$ induces a homotopy automorphism of the above space. Looping down m times and letting $m \to \infty$ we obtain the infinite loop map

$$\mathrm{IND}(X) : Q(P \times_G S(X)_+) \longrightarrow Q(P \times_G S(X)_+).$$

Our next lemma identifies $\mathrm{IND}(X)$ in more computable terms. First some preparations are necessary.

Let W be a representation space for G and $f : S^W \to S^W$ a G-homotopy equivalence. Its G-homotopy class is determined by the set of degrees $\{\deg f^H | H \subseteq G\}$. These degrees (all equal to ± 1) define a unit $d(f)$ of the Burnside ring $A(G)$, cf. [tDP]. Conversely, any $d \in A(G)^\times$ is equal to $d(f)$ for a suitable (W, f).

Given a principal G-bundle $E \to B$ with compact base space there is a map

$$(3.3) \qquad A(G)^\times \longrightarrow [Q(B_+), Q(B_+)]_{\Omega^\infty} \simeq [B_+, Q(B_+)]$$

into the homotopy invertible infinite loop maps. It maps $d(f)$ into the element determined by

$$E \times_G f : E \times_G S^W \longrightarrow E \times_G S^W$$

upon taking fibrewise smash product with $\dot{\xi}$ as above, where ξ is a complement to $E \times_G W$.

In the situation of Theorem 3.3, let $\sigma \in S(X)$ have isotropy group G_σ. The one point compactification of

$$dX_\sigma : T_\sigma F \longrightarrow T_\sigma F$$

defines an element $\chi(X, \sigma) \in A(G_\sigma)^\times$, i.e.

$$(3.4) \qquad \chi(X, \sigma) = \left\{ \det\left(dX_\sigma^H\right) \middle| H \subseteq G_\sigma \right\} \in A(G_\sigma)^\times.$$

We decompose $S(X)$ into its G-orbits,

$$P \times_G S(X) = \coprod P \times_{G_\sigma} \{\sigma\}; \quad \sigma \in S(X)/G.$$

Since $Q(-)$ converts wedge sums into products,

$$Q(P \times_G S(X)_+) = \prod Q(P/G_{\sigma+}); \quad \sigma \in S(X)/G.$$

The image of $\chi(X, \sigma)$ in $[P/G_{\sigma+},\ Q(P/G_{\sigma+})]$ under the map (3.3) is again denoted by $\chi(X, \sigma)$, and we have

Lemma 3.4. *In* $[P \times_G S(X)_+,\ Q(P \times_G S(X)_+]$,

$$\mathrm{IND}(X) = \prod \chi(X, \sigma); \quad \sigma \in S(X)/G.$$

□

Remark. In [BM], Theorem 2.10, the map $\mathrm{IND}(X)$ was falsely asserted to be $\prod \mathrm{ind}_\sigma(X)$ rather than the more complicated expression in Lemma 3.4. The mistake – pointed out in [MP] – occurs at the top of page 141 in [BM]. Indeed the homotopy $\hat{J}_t$ need not be proper. We note that $\mathrm{ind}_\sigma(X) = \chi(X, \sigma)$ if and only if $\det(dX_\sigma^H)$, $H \subseteq G_\sigma$ is independent of H. This happens always if G_σ is of odd order, since $A(G_\sigma)^\times = \{\pm 1\}$.

In our application, the bundle is

$$\pi : EC_q \times_{C_q} F \longrightarrow BC_q$$

where F is the branched cover from Definition 3.1, and X is the vector field from Lemma 3.2. In this case $\chi(X, \sigma) = \mathrm{ind}_\sigma(X)$. Indeed, the action on F is free off the fixed set and it suffices to check that

$$dX_\sigma : S^{T_\sigma F} \longrightarrow S^{T_\sigma F}$$

has equal degrees on fixed sets of the isotropy subgroup $(C_q)_\sigma$. This is clear since $(C_q)_\sigma \neq 1$ only for $\sigma \in \{p_0, \dots, p_k\}$ where dX_σ has degree $+1$. Moreover, the C_q fixed set is S^0, and dX_σ maps it by the identity.

We seek information about the composition

$$\tau_F : B\mathrm{Diff}(F) \xrightarrow{\mathrm{trf}(\pi_F)} Q(E(F)_+) \xrightarrow{Q(T_+^v)} Q\big(BS_+^1\big). \tag{3.5}$$

where $\mathrm{trf}(\pi_F)$ is the transfer of the universal smooth F-bundle, and T^v the classifying map of the vertical tangent bundle of π_F, cf. (2.13). Let F be the C_q-surface of Definition 3.1, and

$$\rho_F : BC_q \longrightarrow B\mathrm{Diff}(F)$$

the associated map. Our next result calculates $\tau_F \circ \rho_F$ in terms of the maps

$$\hat{\psi}^{m_i} : BC_q \xrightarrow{j} BS^1 \xrightarrow{\psi^{m_i}} BS^1 \xrightarrow{\mathrm{incl}} Q\big(BS_+^1\big)$$
$$\hat{\tau}_q : BC_q \xrightarrow{\tau_q} Q((EC_q)_+) \xrightarrow{\simeq} Q(S^0) \hookrightarrow Q\big(BS_+^1\big)$$

where j is the standard map and τ_q is the transfer of the canonical covering $EC_q \to BC_q$.

Theorem 3.5. *The composition $\tau_F \circ \rho_F$ is equal to*

$$\Sigma_{i=0}^k \hat{\psi}^{m_i} + \Sigma\, \varepsilon_j \hat{\tau}_q$$

in $[BC_{q+}, Q(BS_+^1)]$, where ε_j is a sign and the second sum runs over the number of free C_q-orbits in the singular set of the vector field from Lemma 3.2.

Proof. Consider the pull-back diagram

$$\begin{array}{ccc} EC_q \times_{C_q} F & \xrightarrow{\hat{\rho}_F} & E(F) \\ \pi \downarrow & & \pi_F \downarrow \\ BC_q & \xrightarrow{\rho_F} & B\mathrm{Diff}(F). \end{array}$$

Here we can take $EC_q = E\mathrm{Diff}(F) = \mathrm{Emb}(F; \mathbb{R}^{\infty})$. The transfer maps $\mathrm{trf}(\pi)$ and $\mathrm{trf}(\pi_F)$ can thus be defined on all of BC_q and $B\mathrm{Diff}(F)$ respectively, cf. the definition of of the transfer above (2.13). Furthermore, by construction

$$\mathrm{trf}(\pi_F) \circ \rho_F \simeq Q(\hat{\rho}_{F+}) \circ \mathrm{trf}(\pi).^2$$

We can now apply Theorem 3.3 to study π. We have

$$S(X) = \{p_0, \ldots, p_k\} \sqcup S' \times C_q,$$
$$EC_q \times_{C_q} S(X) = (\coprod_{i=0}^{k} BC_q) \sqcup (S' \times EC_q),$$

and hence

$$Q(EC_q \times_{C_q} S(X)_+) = (\prod_{i=0}^{k} Q(BC_{q+})) \times (\prod_{j \in S'} Q(EC_{q+})).$$

The transfer for a sum of covering spaces is the product of the individual transfers. So the transfer for

$$\pi_S : EC_q \times_{C_q} S(X) \longrightarrow BC_q$$

is the product of $(k+1)$ copies of the inclusion $\iota : BC_q \to Q(BC_{q+})$ and $|S'|$ copies of the standard transfer $\tau_q : BC_q \to Q(EC_{q+})$. By (3.2)

$$\hat{\rho}_F^*(T^v E(F)) = EC_q \times_{C_q} TF$$

restricts to $EC_q \times_{C_q} \mathbb{C}(m_i)$ over $EC_q \times_{C_q} \{p_i\}$, and this bundle is classified by $\psi^{m_i} \circ j : BC_q \to BS^1$. An application of Theorem 3.3 and Lemma 3.4 completes the proof. □

3.3. The splitting map μ_p

We use (3.9) and (3.10) on the surfaces constructed in Sect. 3.1. More precisely let us fix a p-adic divisor on S^2,

(3.6) $$D = 1 \cdot p_0 + mp_1 + \ldots + mp_k \in C_0(S^2; \mathbb{Z}_p)$$

with $m \in \mathbb{Z}_p^{\times}$, $1 + km = 0$ and $-k \in \mathbb{Z}$ a topological generator of $\mathbb{Z}_p^{\times}$, or equivalently a generator of $(\mathbb{Z}/p^2)^{\times}$. (If $p = 2$, take $k = 3$.) For each prime power $q = p^n$, let $F = F(n)$ be the closed surface of Definition 3.1 associated with the mod q reduction $D \in C_0(S^2; \mathbb{Z}/q)$. It has genus

$$g(n) = \frac{1}{2}(p^n - 1)(k - 1)$$

---

[2] More generally, transfers commute with pull-backs, at least on finite skeleta.

by the Riemann-Hurwitz formula. We remove two open disks from $F(n)$ to get the surface $F(n)_{1+1}$. There is a commutative diagram

$$\begin{array}{ccc} (BC_{p^n})^{[g(n)-2/2]} & \xrightarrow{\rho_n} & B\mathrm{Diff}(F(n)_{1+1})^+ \\ {\scriptstyle \mathrm{incl}}\downarrow & & {\scriptstyle \mathrm{incl}}\downarrow \\ BC_{p^n} & \xrightarrow{\rho_n} & B\mathrm{Diff}(F(n))^+ \end{array}$$

since the right-hand vertical map is $[(g(n)-2)/2]$-connected by (1.3); $X^{[r]}$ denotes the r-skeleton of X. We can compose the top horizontal map in the above diagram with the map from $B\mathrm{Diff}(F(n)_{1+1})^+$ to $B\Gamma_\infty^+$ to obtain homotopy classes

$$[\rho_n] \in \left[BC_{p^n}^{[g(n)/2]}, B\Gamma_\infty^+\right]. \tag{3.7}$$

We do not know that these elements fit together to define an element of the inverse limit, and hence give a homotopy class

$$\rho_\infty : BC_{p^\infty} \to B\Gamma_\infty^+,$$

with a possible extension over the universal Bockstein $BC_{p^\infty} \to BS^1_{(p)}$. However, we can get around this difficulty upon p-completion.

Recall the notation that $\hat{\psi}^l : BC_{p^n} \to Q(BS^1)$ is the standard map into BS^1 followed by ψ^l and the inclusion into $Q(BS^1)$. Let $\tilde{\tau}_\infty$ denote the composite of $\tau_\infty = \omega_\infty \circ \alpha_\infty$ with the projection of $Q(BS^1_+)$ onto $Q(BS^1)$.

Theorem 3.6. *There exists a map* $\tilde{\mu}_p : BS^1 \to (B\Gamma_\infty^+)^\wedge_p$ *such that*

$$[\tilde{\tau}_\infty \circ \tilde{\mu}_p] = \hat{1} + k\hat{\psi}^{-k} \in \left[BS^1, Q(BS^1)^\wedge_p\right].$$

Proof. Recall from (2.15) that $\tau_{F(n)} \circ \mathrm{incl} \simeq +[2] \circ \tau_\infty \circ \gamma_b$. Thus for the reduced maps, Theorem 3.5 shows that

$$[\tilde{\tau}_\infty \circ \rho_n] = \hat{1} + k\hat{\psi}^{-k} \in \left[BC_{p^n}^{[g(n)/2]}, Q(BS^1)\right].$$

Indeed, since we project into $Q(BS^1)$ the second sum $\Sigma\, \varepsilon_j \hat{\tau}_q$ in Theorem 3.5 disappears. Therefore the subset G_n of $[BC_{p^n}^{[g(n)-2/2]}, B\Gamma_\infty^+]$, given by

$$G_n = \left\{[f] \mid [\tilde{\tau}_\infty \circ f] = \hat{1} + k\hat{\psi}^{-k}\right\}$$

is non-empty. It is also compact, since BC_{p^n} has finite homology in each positive dimension, any of its finite skeleton has a finite homology decomposition, and $B\Gamma_\infty^+$ is of finite type.[3] Then Tychonov's theorem implies that

[3] The homology of the stable mapping class group is finitely generated in each dimension as can be deduced from the finite cell decompositions of moduli spaces in terms of ribbon graphs [Pe], [K], or parallel slit domains [Bö]. An H-space with finitely generated homology in all dimension has also finitely generated homotopy groups in all dimensions.

$\lim G_n \neq \emptyset$. Let $(\rho_n) \in \lim G_n$. Since $g(n) \to \infty$ for $n \to \infty$,

$$\lim\big[BC_{p^n}^{[g(n)/2]}, \big(B\Gamma_\infty^+\big)\big] = \lim\big[BC_{p^n}, \big(B\Gamma_\infty^+\big)\big].$$

We now use Milnor's exact sequence

$$0 \longrightarrow \lim{}^1[\Sigma X_n, E] \longrightarrow [\mathrm{hocolim}\, X_n, E] \longrightarrow \lim[X_n, E] \longrightarrow 0$$

with $X_n = BC_{p^n}^{[g(n)-2/2]}$ and $E = B\Gamma_\infty^+$. The left term vanishes: a similar argument as above shows that $[\Sigma X_k, E]$ is finite; as E is an H-space $[\Sigma X_k, E]$ is a group; the $\lim^1$ of a sequence of finite groups always vanishes, cf. [BK; IX, §3]. Hence,

$$\rho = (\rho_n) : BC_{p^\infty} \longrightarrow B\Gamma_\infty^+$$

is well-defined. The universal (p-local) Bockstein operator

$$\beta_{(p)} : BC_{p^\infty} = B(\mathbb{Q}/\mathbb{Z}_{(p)}) \longrightarrow BS^1_{(p)} = B(B(\mathbb{Z}_{(p)}))$$

with homotopy fibre $B\mathbb{Q}$ induces an isomorphism on ordinary homology with $\mathbb{Z}/p$ coefficients. Also the map $BS^1 \to (BS^1)^\wedge_p$ induces an isomorphism in $\mathbb{Z}/p$-homology. The induced map

$$\big[BS^1, E^\wedge_p\big] \xleftarrow{\simeq} \big[(BS^1)^\wedge_p, E^\wedge_p\big] \xrightarrow{\beta^*_{(p)}} \big[BC_{p^\infty}, E^\wedge_p\big]$$

is an isomorphism, cf. [BK; VI, Prop. 5.4], and $\tilde\mu_p = \beta^*_{(p)}(\rho) = \beta^*_{(p)}((\rho_n))$ is well-defined. □

Proof of Theorem 1.2. The map $\tilde\mu_p$ of Theorem 3.6 together with the map $S^0 \to \mathbb{Z} \times B\Gamma^+_\infty$ that sends the non base point into $(1, *)$ extends to a map

$$Q\big(BS^1_+\big)^\wedge_p \xrightarrow{\mu_p} \big(\mathbb{Z} \times B\Gamma^+_\infty\big)^\wedge_p$$

via the infinite loop space structure on $\mathbb{Z} \times B\Gamma^+_\infty$. We study

$$\tau_\infty \circ \mu_p : Q\big(BS^1_+\big)^\wedge_p \longrightarrow Q\big(BS^1_+\big)^\wedge_p.$$

The composition

$$S^0 \longrightarrow \mathbb{Z} \times B\Gamma^+_\infty \xrightarrow{\tau_\infty} Q\big(BS^1_+\big) = Q(S^0) \times Q(BS^1)$$

is the map that sends the non base point to $(-2, *)$ by (2.14). Since $\tau_\infty = \omega_\infty \circ \alpha_\infty$ is an infinite loop map the composite

$$Q(S^0) \longrightarrow \mathbb{Z} \times B\Gamma^+_\infty \xrightarrow{\tau_\infty} Q(S^0) \times Q(BS^1)$$

has first component multiplication by -2 and trivial second component. Apply Theorem 3.6. □

4. Proofs of the splitting theorems

In this section p is an odd prime.

4.1. Splitting of $Q(BS^1_+)^\wedge_p$

We consider the (reduced) cohomology theory associated to $Q(BS^1_+)^\wedge_p$,

$$E^i(X) = \big[X, Q\big(S^i \wedge BS^1_+\big)^\wedge_p\big].$$

Recall, for $k \in \mathbb{Z}_p^\times$, that $\psi^k : (BS^1)^\wedge_p \to (BS^1)^\wedge_p$ represents the cohomology class $k.c_1(L)$. Then $Q(1 \wedge \psi^k)$ defines a natural endomorphisms of $E^i(X)$ that commutes with suspension. Let $\omega : \mathbb{Z}/p^\times \to \mathbb{Z}_p^\times$ be the Teichmüller character that splits the natural projection $\mathbb{Z}_p^\times \to \mathbb{Z}/p^\times$. We get a natural action of $\mathbb{Z}_p[\mathbb{Z}/p^\times]$ on $E^*(X)$, where $l \in \mathbb{Z}/p^\times$ acts on $E^*(X)$ via the map

$$\psi^{\omega(l)} : \big(BS^1_+\big)^\wedge_p \longrightarrow \big(BS^1_+\big)^\wedge_p.$$

The ring $\mathbb{Z}_p[\mathbb{Z}/p^\times]$ is semisimple and decomposes into a sum of $(p-1)$ copies of $\mathbb{Z}_p$, and there is an induced isomorphism of cohomology theories

$$E^*(X) \cong E^*_0(X) \oplus \ldots \oplus E^*_{p-2}(X). \tag{4.1}$$

More precisely, if l generates $\mathbb{Z}/p^\times$ then

$$e_i = \frac{1}{p-1}\sum_{\nu=0}^{p-2} \omega(l^{-\nu i})l^\nu, \quad i = 0, \ldots, p-2 \tag{4.2}$$

are orthogonal idempotents of $\mathbb{Z}_p[\mathbb{Z}/p^\times]$, and

$$E^*_i(X) = e_i E^*(X).$$

The idempotents e_i also define a splitting of p-complete K-theory, often called the Adams' splitting [A]. We get from (4.1), and its K-theory analogue, the induced splitting of infinite loop spaces

$$\begin{aligned} Q\big(BS^1_+\big)^\wedge_p &\simeq \Omega^\infty E_0 \times \Omega^\infty E_1 \times \ldots \times \Omega^\infty E_{p-2} \\ (\mathbb{Z} \times B\mathrm{U})^\wedge_p &\simeq B_0 \times B_1 \times \ldots \times B_{p-2}. \end{aligned} \tag{4.3}$$

The ψ^k-operations reduce to the identity on the factor $Q(S^0)$ in $Q(BS^1_+) = Q(S^0) \times Q(BS^1)$, so

$$\Omega^\infty E_0 = Q(S^0)_{(p)} \times \Omega^\infty \tilde{E}_0$$

and

$$Q(BS^1)^\wedge_p = \Omega^\infty \tilde{E}_0 \times \Omega^\infty \tilde{E}_1 \times \cdots \times \Omega^\infty \tilde{E}_{p-2},$$

with $\tilde{E}_i = E_i$ for $i > 0$.

The canonical line bundle induces a map from BS^1_+ into the 1-component of $\mathbb{Z} \times B\mathrm{U}$ that extends to a map $\hat{L}$ of $Q(BS^1_+)$ to $\mathbb{Z} \times B\mathrm{U}$ by Bott periodicity, cf. (1.8). If we identify $Q(BS^1_+)$ with $QS^0 \times Q(BS^1)$ via the natural projections then $\hat{L}$ becomes identified with (deg, $(L-1)^\wedge$). In particular we have on the zero component

$$\hat{L} \simeq (L-1)^\wedge : Q_0\big(BS^1_+\big) \longrightarrow B\mathrm{U}. \tag{4.4}$$

Both $\hat{L}$ and $(L-1)^\wedge$ commute with Adams operations, and hence the action of the idempotents e_i, and define infinite loop maps from $\Omega^\infty E_i$ to B_i, and $\Omega^\infty \tilde{E}_i$ to $\tilde{B}_i$.

Lemma 4.1. *Let p be an odd prime and $g \in \mathbb{Z}_p^\times$ a topological generator. Then*

$$1 - g\psi^g : \Omega^\infty \tilde{E}_i \longrightarrow \Omega^\infty \tilde{E}_i$$

is a homotopy equivalence for $i = 0, 1, \ldots, p-3$, but not for $i = p-2$.

Proof. It suffices to prove that the induced map on spectrum homology

$$(1 - g\psi^g)_* : H_*^{\mathrm{spec}}(\tilde{E}_i) \longrightarrow H_*^{\mathrm{spec}}(\tilde{E}_i)$$

is an isomorphism for each $i \neq p-2$. The homology of $\tilde{E}_i$ is one copy of $\mathbb{Z}_p$ in each degree $2n$ with $n \equiv i (\mathrm{mod}\, p-1)$ and zero in other degrees. Indeed, the wedge product $\tilde{E}_0 \vee \ldots \vee \tilde{E}_{p-2}$ is the p-complete suspension spectrum of BS^1, so

$$H_*^{\mathrm{spec}}\big(\tilde{E}_0 \vee \cdots \vee \tilde{E}_{p-2}\big) = H_*^{\mathrm{spec}}\big((\Sigma^\infty BS^1)^\wedge_p\big) = H_*\big(BS^1; \mathbb{Z}_p\big),$$

a copy of $\mathbb{Z}_p$ in each even degree. On the other hand, it follows from (4.2) that

$$\psi^{\omega(l)} \circ e_i = \omega(l^i) e_i,$$

so that $\psi^{\omega(l)}$ induces multiplication by $\omega(l^i)$ on $H_{2n}^{\mathrm{spec}}(\tilde{E}_i)$ for all n. But $\psi^{\omega(l)}$ induces multiplication by $\omega(l)^n$ on $H_{2n}(BS^1; \mathbb{Z}_p)$. Thus $\omega(l^n) = \omega(l^i)$ on $H_{2n}^{\mathrm{spec}}(\tilde{E}_i)$, and $n \equiv i (\mathrm{mod}\, p-1)$.

The map $1 - g\psi^g$ induces multiplication by $1 - g^{n+1}$ on $H_{2n}(BS^1; \mathbb{Z}_p)$ and $1 - g^{n+1}$ is a p-adic unit precisely when $n \not\equiv -1 (\mathrm{mod}\, p-1)$. □

As an immediate consequence of Theorem 1.2 and Theorem 2.5 we have:

Theorem 4.2. *The map*

$$\big(\mathbb{Z} \times B\Gamma^+_\infty\big)^\wedge_p \xrightarrow{\tau_\infty} Q\big(BS^1_+\big)^\wedge_p \xrightarrow{\mathrm{proj}} \Omega^\infty E_0 \times \ldots \times \Omega^\infty E_{p-3},$$

with $\Omega^\infty E_0 = Q(S^0)_{(p)} \times \Omega^\infty \tilde{E}_0$, is split surjective as a map of infinite loop spaces.

4.2. Old and new homology of $B\Gamma_\infty^+$

We first describe the relationship between the classes $\kappa_i \in H^{2i}(B\Gamma_\infty^+;\mathbb{Q})$, cf. (1.4), and the map $\tau_\infty = \omega_\infty \circ \alpha_\infty$ from $\mathbb{Z}\times B\Gamma_\infty^+$ to $Q(BS_+^1)$. Restricting to the zero component and projecting onto $Q(BS^1)$ we get

$$\tilde{\tau}_\infty : B\Gamma_\infty^+ \longrightarrow Q(BS^1).$$

We compose with $(L-1)^\wedge$, cf. (4.4).

Theorem 4.3. *The composition*

$$(L-1)^\wedge \circ \tilde{\tau}_\infty : B\Gamma_\infty^+ \longrightarrow B\mathrm{U}$$

in cohomology maps the integral Chern character class $i!ch_i$ to κ_i.

Proof. Let $i : BS^1 \to Q(BS^1)$ be the inclusion. The image of

$$(L-1)_*^\wedge \circ i_* : H_*(BS^1;\mathbb{Z}) \longrightarrow H_*(B\mathrm{U};\mathbb{Z})$$

define polynomial generators a_i in degree $2i$ of $H_*(B\mathrm{U};\mathbb{Z})$, and $i!ch_i$ is the unique primitive element of the Hopf algebra $H^*(B\mathrm{U};\mathbb{Z})$ with $< i!ch_i, a_i >= 1$. Denote also by a_i the corresponding element of $H_*(Q(BS^1);\mathbb{Z})$. Since

$$(L-1)_*^\wedge : H_*(Q(BS^1);\mathbb{Z})/\text{torsion} \xrightarrow{\simeq} H_*(B\mathrm{U};\mathbb{Z}),$$

we get in particular that

$$((L-1)^\wedge)^*(i!ch_i) \in H^{2i}(Q(BS^1);\mathbb{Q})$$

is the unique primitive element that evaluates to $+1$ on a_i. The cohomology suspension,

$$\sigma^* : H^*(BS^1;\mathbb{Z}) \longrightarrow H^*(Q(BS^1);\mathbb{Z}),$$

induced from the evaluation maps $S^s \wedge \Omega^s S^s(BS^1) \to S^s \wedge BS^1$, sends $x^i \in H^{2i}(BS^1;\mathbb{Q})$, $x = c_1(L)$, into a primitive element with the same evaluation property, so

$$\sigma^*(x^i) = ((L-1)^\wedge)^*(i!ch_i).$$

In view of (2.13) and (2.15), it suffices to show that the transfer map of (1.6) composed with the classifying map T^v for T^vE,

$$B_+ \xrightarrow{\mathrm{trf}(\pi)} Q(E_+) \xrightarrow{Q(T_+^v)} Q\big(BS_+^1\big),$$

pulls back $\sigma^*(x^i)$ to κ_i of (1.1), or equivalently that the adjoint

$$\Sigma^\infty B_+ \longrightarrow \Sigma^\infty E_+$$

maps $c_1(T^v(E)^i)$ to κ_i. This follows because we have the factorization

$$S^k \wedge B_+ \longrightarrow \mathrm{Th}(N^v E) \xrightarrow{\omega} \mathrm{Th}(N^v E \oplus T^v E) = S^k \wedge E_+$$

from (1.5) and because

$$\omega^*(\Sigma^k(\alpha)) = U.\alpha.c_1(T^v E), \qquad \alpha \in H^*(E)$$

where U is the Thom class of $\mathrm{Th}(N^v E)$. □

Corollary 4.4. *The map $\alpha_\infty : \mathbb{Z} \times B\Gamma^+_\infty \to \Omega^\infty \mathbb{C}P^\infty_{-1}$ is an isomorphism on homotopy groups in dimensions $* \leq 2$.*

Proof. Recall the fibration (1.7)

$$\Omega^\infty \mathbb{C}P^\infty_{-1} \xrightarrow{\omega_\infty} Q\big(BS^1_+\big) \xrightarrow{\partial} Q(S^{-1}).$$

Using known values of $\pi_*(Q_0(S^0))$ in low dimensions and the fact that the restriction of ∂ to QS^0 is induced from the stable Hopf map η one gets

$$\pi_i\big(\Omega^\infty \mathbb{C}P^\infty_{-1}\big) = \mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}/24 \quad \text{for} \quad i = 0, 1, 2, 3,$$

and hence the first non-zero homology groups:

(4.5) $$H_i\big(\Omega^\infty \mathbb{C}P^\infty_{-1}\big) = \mathbb{Z}, \mathbb{Z}/24 \quad \text{for} \quad i = 2, 3.$$

By Lemma 2.6, α_∞ is an isomorphism on components, and as both spaces have simply connected components it is trivally an isomorphism on fundamental groups. In degree 2 the homotopy exact sequence of the fibration of zero components,

$$0 \longrightarrow \pi_2\big(\Omega^\infty_0 \mathbb{C}P^\infty_{-1}\big) \longrightarrow \pi_2\big(Q_0\big(BS^1_+\big)\big) \longrightarrow \pi_3\big(Q_0(S^0)\big) \longrightarrow 0$$

is

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

and the middle $\mathbb{Z}/2$, generated by η^2, maps non-trivially. Hence

$$\pi_2\big(\Omega^\infty_0 \mathbb{C}P^\infty_{-1}\big) \xrightarrow{\omega_\infty} \pi_2\big(Q_0\big(BS^1_+\big)\big) \xrightarrow{\hat{L}} \pi_2(B\mathrm{U})$$

is multiplication by 12, and $\frac{1}{12}\omega^*_\infty \hat{L}^*(c_1) \in H^2(\Omega^\infty_0 \mathbb{C}P^\infty_{-1})$ is the generator.

Since $H^2(B\Gamma_\infty) = \mathbb{Z}$ it follows from Proposition 4.3 that $H^2(\alpha_\infty)$ is an isomorphism if and only if $\kappa_i \in H^2(B\Gamma_\infty)$ is divisible precisely by 12. This follows from [H1]. Indeed Harer describes the generator as the homomorphism

$$\Omega_2(B\Gamma_\infty) \longrightarrow \mathbb{Z}$$

that to a bordism class $[M^2 \xrightarrow{f} B\Gamma_\infty]$ associates $\frac{1}{4}$sign (W) where $F \to W \xrightarrow{\pi} M^2$ is the induced surface bundle and sign(W) is the signature of W. The index formula gives

$$\text{sign }(W) = <\frac{1}{3}p_1(TW), [W]> = \frac{1}{3} < \int_F p_1(TW), [M] > .$$

Since $TW = T^vW \oplus \pi^*TM$ is the sum of two complex line bundles,

$$p_1(TW) = c_1(T^vW)^2 + \pi^* c_1(TM)^2 = c_1(T^vW)^2$$

and $\int_F p_1(TW) = \kappa_1$, cf. (1.1). This completes the argument. □

The homological structure of $Q(BS^1_+)$ is completely known. The original source is [DL], but [CLM] might be a better reference.

Consider sequences $I = (\varepsilon_1, s_1, \ldots, \varepsilon_k, s_k)$ of non-negative integers with

$$\varepsilon_j \in \{0, 1\}, \quad s_j \geq \varepsilon_j, \quad ps_j - \varepsilon_j \geq s_{j-1},$$

and define

$$e(I) = 2s_1 - \varepsilon_1 - \sum_{j=2}^{k}(2s_j(p-1) - \varepsilon_j),$$

$$b(I) = \varepsilon_1,\ d(I) = \sum_{j=1}^{k}(2s_j(p-1) - \varepsilon_j).$$

For any infinite loop space X and each I there is a homology operation

$$Q^I : H_q(X; \mathbb{Z}/p) \longrightarrow H_{q+d(I)}(X; \mathbb{Z}/p)$$

which can be non-zero only if $e(I) + b(I) \geq q$. The homology of $Q(BS^1_+)$ can be described in terms of the homology operations, applied to $H_*(BS^1; \mathbb{Z}/p) \subset H_*(Q(BS^1_+); \mathbb{Z}/p)$. Indeed, if

$$T = \left\{Q^I a_q \;\middle|\; q \geq 0,\ e(I) + b(I) > 2q\right\},$$

then

$$(4.6) \qquad H_*\big(Q\big(BS^1_+\big); \mathbb{Z}/p\big) = \mathbf{S}_*(T) \otimes \mathbb{Z}/p[\mathbb{Z}],$$

where $\mathbf{S}_*(T)$ denotes the free graded commutative algebra generated by T, i.e. a tensor product of the polynomial algebra on the even dimensional generators and the exterior algebra on the odd dimensional generators, cf. [CLM; p. 42]. By definition

$$H_*\big(\Omega^\infty E_0 \times \ldots \Omega^\infty E_{p-3}\big) = (1 - e_{p-2})_* H_*\big(Q\big(BS^1_+\big)\big),$$

and Theorem 4.2 together with (4.6) proves

Corollary 4.5. *The mod p homology $H_*(B\Gamma_\infty^+;\mathbb{Z}/p)$ contains the free commutative algebra $\mathbf{S}_*(T')$ where*

$$T' = \{Q^I a_q \in T \mid q \neq -1 \ (\mathit{mod}\ p-1)\}. \qquad \square$$

The Bockstein structure of $H_*(Q(BS_+^1);\mathbb{Z}/p)$ is completely known, see e.g. Theorem 4.13 of [CLM; p. 49]. This implies the structure of $H_*(Q(BS_+^1);\mathbb{Z}_p)$, and hence $(1-e_{p-2})_* H_*(Q(BS_+^1);\mathbb{Z}_p)$, giving a huge collection of new p-torsion classes in $H_*(B\Gamma_\infty^+;\mathbb{Z})$.

4.3. Formula (1.11) and the proof of Theorem 1.3

The proof is based upon results from [MS] where the fibration sequence of (1.7),

$$\Omega^\infty \mathbb{C}P_{-1}^\infty \xrightarrow{\omega_\infty} Q(BS_+^1) \xrightarrow{\partial} Q(S^{-1}),$$

is examined in considerable detail after localization (or completion) at any prime p.

The bundle $-L_s$ over $\mathbb{C}P^s$ has complex dimension s. Let λ_{-L_s} denote the standard K-theory Thom class. The elements

$$L_s \lambda_{-L_s} \in \tilde{K}(\mathrm{Th}(-L_s))$$

fit together to define a map

$$\eta' : \Omega^\infty \mathbb{C}P_{-1}^\infty \longrightarrow \mathbb{Z} \times B\mathrm{U}.$$

Diagram (6.8) of [MS] implies for every prime p the following map of fibration sequences

(4.7)
$$\begin{array}{ccccccc}
Q(S^{-2}) & \longrightarrow & \Omega^\infty \mathbb{C}P_{-1}^\infty & \longrightarrow & Q(BS_+^1) & \longrightarrow & Q(S^{-1}) \\
\Big\downarrow{\scriptstyle \Omega^2(e_\mathbb{C})} & & \Big\downarrow{\scriptstyle \eta'} & & \Big\downarrow{\scriptstyle l'} & & \Big\downarrow{\scriptstyle \Omega(e_\mathbb{C})} \\
\Omega^2(\mathrm{im} J_\mathbb{C}) & \longrightarrow & (\mathbb{Z}\times B\mathrm{U})_{(p)} & \xrightarrow{1-g\psi^g} & (\mathbb{Z}\times B\mathrm{U})_{(p)} & \longrightarrow & \Omega(\mathrm{im} J_\mathbb{C}).
\end{array}$$

As in previous sections $g \in \mathbb{Z}$ is chosen so that it generates $(\mathbb{Z}/p^2)^\times$ multiplicatively when $p > 2$ and $g = 3$ when $p = 2$. The space $\mathrm{im} J_\mathbb{C}$ is the homotopy fiber of $1 - \psi^g$ acting on $\mathbb{Z} \times B\mathrm{U}$, and l' is the composition

$$l' : Q(BS_+^1) \xrightarrow{\hat{L}} \mathbb{Z} \times B\mathrm{U} \xrightarrow{\Omega^2(\rho^g - 1)} (\mathbb{Z} \times B\mathrm{U})_{(p)}$$

where

$$\rho^g : K(X) \longrightarrow 1 + \tilde{K}(X;\mathbb{Z}_{(p)})$$

is the “cannibalistic characteristic class” that on a line bundle L is given by

$$\rho^g(L) = \frac{1}{g}\left(\frac{L^g - 1}{L - 1}\right).$$

Remark. Let $F \to E \to B$ be a smooth oriented surface bundle classified by $f : B \to B\Gamma_\infty$. Then $\eta' \circ \alpha_\infty \circ f : B \to \mathbb{Z} \times B\mathrm{U}$ classifies the element $\pi_!(T^v E)$ where $\pi_! : K(E) \to K(B)$ is the push forward map with respect to the standard Thom class and $T^v E$ is viewed as a complex line bundle.

Let B_i be the i-th Bernoulli number in the classical notation used in [A2]; $B_1 = \frac{1}{6}$.

Theorem 4.6. *In $H^*(B\Gamma_\infty; \mathbb{Q})$,*

$$(\eta' \circ \alpha_\infty)^*((2i-1)!\, ch_{2i-1}) = (-1)^i \frac{B_i}{2i} \kappa_{2i-1}.$$

Proof. Let p be any prime and choose g as above. It follows from Theorem 5.18 of [A2] that

$$\pi_{4i}(\rho^g - 1) : \pi_{4i}(B\mathrm{U}) \longrightarrow \pi_{4i}(B\mathrm{U}_{(p)})$$

multiplies by $(-1)^{i-1}(g^{2i}-1)\frac{B_i}{2i}$. Since

$$\pi_{4i-2}(1 - g\psi^g) : \pi_{4i-2}(B\mathrm{U}) \longrightarrow \pi_{4i-2}(B\mathrm{U}_{(p)})$$

multiplies by $1 - g^{2i}$, diagram (4.7) and Theorem 4.3 give the stated equation multiplied by $(1 - g^{2i})$, and hence the required formula. □

Remark. It appears that the family index theorem for the fibrewise $\bar\partial$ operator on a smooth oriented surface bundle will show that $\eta \simeq \eta' \circ \alpha_\infty$, where

$$\eta : B\Gamma_\infty^+ \longrightarrow B\mathrm{Sp}(\mathbb{R})$$

(cf. 1.11). The map η factors through $B\mathrm{Sp}(\mathbb{Z})^+$. We do not know at present if η' factors through $B\mathrm{Sp}(\mathbb{Z})^+$, but hope to return to these questions in the future.

The maps η' and l' are not split surjections, not even rationally. However, in Theorem 6.3 of [MS] it was shown how to rechoose the maps when the prime p is odd to get a diagram with split surjective maps:

$$(4.8)\qquad \begin{array}{ccccc} \Omega^\infty\left(\mathbb{C}P^\infty_{-1}\right)^\wedge_p & \xrightarrow{\omega_\infty} & Q\left(BS^1_+\right)^\wedge_p & \xrightarrow{\partial} & Q(S^{-1})^\wedge_p \\ \downarrow{\scriptstyle l_{-1}} & & \downarrow{\scriptstyle l_0} & & \downarrow{\scriptstyle \Omega(e_\mathbb{C})} \\ (\mathbb{Z} \times B\mathrm{U})^\wedge_p & \xrightarrow{1-g\psi^g} & (\mathbb{Z} \times B\mathrm{U})^\wedge_p & \longrightarrow & \Omega(\mathrm{im} J)^\wedge_p \end{array}$$

It is easy to see that the $(2s+2)$-dimensional skeleton of $\mathrm{Th}(-L_s)$ is the mapping cone of the Hopf map $\eta : S^{2s+1} \to S^{2s}$ which for $s > 1$ has order 2. At odd primes there results a map from S^{2s+2} into $\mathrm{Th}(-L_s)$, and hence

$$Q(S^0)_{(p)} \longrightarrow \left(\Omega^\infty \mathbb{C}P^\infty_{-1}\right)_{(p)}.$$

As argued before in the proof of Lemma 2.6, the composition

$$Q(S^0)_{(p)} \longrightarrow \left(\Omega^\infty \mathbb{C}P^\infty_{-1}\right)_{(p)} \xrightarrow{\omega_\infty} Q\left(BS^1_+\right)_{(p)} \xrightarrow{\mathrm{proj}_1} Q(S^0)_{(p)}$$

is twice the identity. We divide out the factor $Q(S^0)_{(p)}$ to get a reduced version of the above fibration,

$$\left(\Omega^\infty \mathbb{C}P^\infty_{-1}/S^0\right)_{(p)} \xrightarrow{\tilde{\omega}_\infty} Q(BS^1)_{(p)} \xrightarrow{\partial} Q(S^{-1})_{(p)}.$$

We have the following reduced version of (4.8).

(4.9)
$$\begin{array}{ccccc} \Omega^\infty\left(\mathbb{C}P^\infty_{-1}/S^0\right)^\wedge_p & \xrightarrow{\tilde{\omega}_\infty} & Q(BS^1)^\wedge_p & \longrightarrow & Q(S^{-1})^\wedge_p \\ \downarrow{\scriptstyle \tilde{l}_{-1}} & & \downarrow{\scriptstyle \tilde{l}_0} & & \downarrow{\scriptstyle (\Omega e_{\mathbb{C}})} \\ B\mathrm{U}^\wedge_p & \xrightarrow{1-g\psi^g} & B\mathrm{U}^\wedge_p & \longrightarrow & \Omega(\mathrm{im}J)^\wedge_p. \end{array}$$

Here $\tilde{l}_0$ is $(L-1)^\wedge$ composed with $\Omega^2(h)$, where

$$h = e_0(\rho^g - 1) + (1 - e_0)(1 - \psi^g)$$

with e_0 the idempotent considered in Theorem 4.1. The map h is a homotopy equivalence. This follows by the results of [A2] used in the proof of Theorem 6.4.

It is well-known that

$$(L-1)^\wedge : Q(BS^1) \longrightarrow B\mathrm{U}$$

is split by a map

$$\lambda : B\mathrm{U} \longrightarrow Q(BS^1)$$

(see [S2] or [MS, §5, §7]). The map λ can be delooped once but is not an infinite loop map. It has the property that

$$BS^1 \xrightarrow{L-1} B\mathrm{U} \xrightarrow{\lambda} Q(BS^1)$$

is homotopic to the standard inclusion.

Lemma 4.7. *The composition*

$$\varphi^g : B\mathrm{U}^\wedge_p \xrightarrow{\lambda} Q(BS^1)^\wedge_p \xrightarrow{Q(\psi^g)} Q(BS^1)^\wedge_p \xrightarrow{(L-1)^\wedge} B\mathrm{U}^\wedge_p$$

is homotopic to ψ^g.

Proof. The diagram

$$\begin{array}{ccccccc}
BU^\wedge_p & \xrightarrow{\lambda} & Q(BS^1)^\wedge_p & \xrightarrow{Q(\psi^g)} & Q(BS^1)^\wedge_p & \xrightarrow{(L-1)^\wedge} & BU^\wedge_p \\
{\scriptstyle L-1}\uparrow & & {\scriptstyle \mathrm{incl}}\uparrow & & {\scriptstyle \mathrm{incl}}\uparrow & & {\scriptstyle L-1}\uparrow \\
BS^1 & = & BS^1 & \xrightarrow{\psi^g} & (BS^1) & = & (BS^1)
\end{array}$$

is homotopy commutative. Since the image of $H_*(BS^1;\mathbb{Z})$ in $H_*(BU;\mathbb{Z})$ under $(L-1)_*$ is a set of algebra generators, and φ^g is a loop map

$$\psi^g = \varphi^g : H_*(BU;\mathbb{Z}) \longrightarrow H_*(BU;\mathbb{Z}).$$

The obvious map

(4.10) $$[BU, BU] \longrightarrow \mathrm{Hom}(H_*(BU;\mathbb{Q}); H_*(BU;\mathbb{Q}))$$

is injective (see e.g. Lemma 7.3 of [MS]), so $\psi^g \simeq \varphi^g$ as claimed. □

We now complete the proof of Theorem 1.3 by showing that the composition

$$\beta : BU^\wedge_p \xrightarrow{\lambda} Q(BS^1)^\wedge_p \xrightarrow{\hat{\mu}_p} \left(B\Gamma^+_\infty\right)^\wedge_p \xrightarrow{\tilde{\alpha}_\infty} \Omega^\infty\left(\mathbb{C}P^\infty_{-1}/S^0\right)^\wedge_p \xrightarrow{\tilde{l}_{-1}} BU^\wedge_p$$

is a homotopy equivalence. This in turn is a consequence of the homotopy commutative diagram

$$\begin{array}{ccccc}
\left(B\Gamma^+_\infty\right)^\wedge_p & \xrightarrow{\tilde{\alpha}_\infty} & \Omega^\infty\left(\mathbb{C}P^\infty_{-1}/S^0\right)^\wedge_p & \xrightarrow{\tilde{l}_{-1}} & BU^\wedge_p \\
{\scriptstyle \hat{\mu}_p}\uparrow & & {\scriptstyle \tilde{\omega}_\infty}\downarrow & & {\scriptstyle 1-g\psi^g}\downarrow \\
Q(BS^1)^\wedge_p & \xrightarrow{1-g\psi^g} & Q(BS^1)^\wedge_p & \xrightarrow{\tilde{l}_0} & BU^\wedge_p \\
{\scriptstyle \lambda}\uparrow & & {\scriptstyle (L-1)^\wedge}\downarrow & & \Vert \\
BU^\wedge_p & \xrightarrow{1-g\psi^g} & BU^\wedge_p & \xrightarrow{\Omega^2(h)} & BU^\wedge_p,
\end{array}$$

which shows that $(1-g\psi^g)\circ\beta \simeq \Omega^2(h)\circ(1-g\psi^g)$. Since $(1-g\psi^g)$ is non-zero on all homotopy groups, β and $\Omega^2(h)$ induce the same map on homotopy groups. It follows that β and $\Omega^2(h)$ induce the same map on rational homology and so are homotopic by (4.10). Since $\Omega^2(h)$ is a homotopy equivalence, so is β. □

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