# THE STABLE MAPPING CLASS GROUP AND STABLE HOMOTOPY THEORY

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Abstract. This overview is intended as a lightweight companion to the long article [20]. One of the main results there is the determination of the rational cohomology of the stable mapping class group, in agreement with the Mumford conjecture [26]. This is part of a recent development in surface theory which was set in motion by Ulrike Tillmann's discovery [34] that Quillen's plus construction turns the classifying space of the stable mapping class group into an infinite loop space. Tillmann's discovery depends heavily on Harer's homological stability theorem [15] for mapping class groups, which can be regarded as one of the high points of geometric surface theory.

## 1. Surface bundles without stable homotopy theory

We denote by $F_{g,b}$ an oriented smooth compact surface of genus $g$ with $b$ boundary components; if $b = 0$, we also write $F_g$. Let $\mathrm{Diff}(F_{g,b};\partial)$ be the topological group of all diffeomorphisms $F_{g,b} \to F_{g,b}$ which respect the orientation and restrict to the identity on the boundary. (This is equipped with the Whitney $C^\infty$ topology.) Let $\mathrm{Diff}_1(F_{g,b};\partial)$ be the open subgroup consisting of those diffeomorphisms $F_{g,b} \to F_{g,b}$ which are homotopic to the identity relative to the boundary.

**Theorem 1.1.** [10], [11]. *If $g > 1$ or $b > 0$, then $\mathrm{Diff}_1(F_{g,b};\partial)$ is contractible.*

*Idea of proof.* For simplicity suppose that $b = 0$, hence $g > 1$. Write $F = F_g = F_{g,0}$. Let $\mathcal{H}(F)$ be the space of hyperbolic metrics (i.e., Riemannian metrics of constant sectional curvature $-1$) on $F$. The group $\mathrm{Diff}_1(F;\partial)$ acts on $\mathcal{H}(F)$ by transport of metrics. The action is free and the orbit space is the Teichmüller space $\mathcal{T}(F)$. The projection map

$$\mathcal{H}(F) \longrightarrow \mathcal{T}(F)$$

admits local sections, so that $\mathcal{H}(F)$ is the total space of a principal bundle with structure group $\mathrm{Diff}_1(F;\partial)$. By Teichmüller theory, $\mathcal{T}(F)$ is homeomorphic to a euclidean space, hence contractible. It is therefore enough to show that $\mathcal{H}(F)$ is contractible.

This is not easy. Let $\mathcal{S}(F)$ be the set of conformal structures on $F$ (equivalently, complex manifold structures on $F$ which refine the given smooth structure and are compatible with the orientation of $F$). Let $\mathcal{J}(F)$ be the set of almost complex structures on $F$. Elements of $\mathcal{J}(F)$ can be regarded as smooth vector bundle automorphisms $J\colon TF \to TF$ with the property $J^2 = -\mathrm{id}$ and $\det(a(v), aJ(v)) > 0$ for any $x \in F$, $v \in T_xF$ and oriented isomorphism $a\colon T_xF \to \mathbb{R}^2$. Hence $\mathcal{J}(F)$ has

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a canonical (Whitney $C^\infty$) topology. It is a consequence of the "uniformization theorem" that the forgetful map $\mathcal{H}(F) \to \mathcal{S}(F)$ is a bijection. The forgetful map $\mathcal{S}(F) \to \mathcal{J}(F)$ is also a bijection. This is another hard old theorem (the Korn–Lichtenstein theorem); see e.g. [8], [27]. Hence the composite map $\mathcal{H}(F) \to \mathcal{J}(F)$ is a bijection. It is clearly continuous. One of the main points of [10] and [11] is that the inverse $\mathcal{J}(F) \to \mathcal{H}(F)$ is also continuous. Hence $\mathcal{H}(F)$ is homeomorphic to $\mathcal{J}(F)$, and $\mathcal{J}(F)$ is clearly contractible. □

**Definition 1.2.** With the assumptions of theorem 1.1, the *mapping class group* $\Gamma_{g,b}$ is $\pi_0\mathrm{Diff}(F_{g,b};\partial) = \mathrm{Diff}(F_{g,b};\partial)/\mathrm{Diff}_1(F_{g,b};\partial)$.

**Remark 1.3.** $B\mathrm{Diff}(F_{g,b};\partial) \simeq B\Gamma_{g,b}$.

*Proof.* By theorem 1.1, the projection $\mathrm{Diff}(F_{g,b};\partial) \to \Gamma_{g,b}$ is a homotopy equivalence. Hence the induced map $B\mathrm{Diff}(F_{g,b};\partial) \longrightarrow B\Gamma_{g,b}$ is a homotopy equivalence. □

It seems that a homological theory of mapping class groups emerged only after the Earle–Eells–Schatz result, theorem 1.1. One of the most basic homological results is the following, due to Powell [29].

**Proposition 1.4.** *$H_1(B\Gamma_g;\mathbb{Z}) = 0$ for $g \geq 3$.*

This is of course equivalent to the statement that $\Gamma_g$ is perfect when $g \geq 3$. The proof is based on a result of Dehn's which states that $\Gamma_g$ can be generated by a finite selection of Dehn twists along simple closed curves in $F_g$. Powell shows that each of these generating Dehn twists is a commutator.
An important consequence of proposition 1.4 is that there exist a simply connected space $B\Gamma_g^+$ and a map $f\colon B\Gamma_g \to B\Gamma_g^+$ which induces an isomorphism in integer homology. The space $B\Gamma_g^+$ and the map $f$ are essentially unique and the whole construction is a special case of Quillen's *plus construction*, beautifully explained in [1].

Around 1980, Hatcher and Thurston [16] succeeded in showing that $\Gamma_g$ is finitely presented. The new ideas introduced in their paper may have influenced the proof of the following theorem. (*Added later*: This refers to the "curve complexes" used by Hatcher and Thurston, but it seems that W. J. Harvey invented them a few years before that.)

**Theorem 1.5.** *Let $N$ be an oriented compact surface, $N = N_1 \cup N_2$ where $N_1 \cap N_2$ is a union of finitely many smooth circles in $N \smallsetminus \partial N$. Suppose that $N_1 \cong F_{g,b}$ and $N \cong F_{h,c}$. Then the homomorphism $H_*(B\Gamma_{g,b};\mathbb{Z}) \to H_*(B\Gamma_{h,c};\mathbb{Z})$ induced by the inclusion $N_1 \to N$ is an isomorphism for $* \leq g/2 - 1$.*

This is the homological stability theorem of Harer [15] with improvements due to Ivanov [17], [18]. It is a hard theorem and we shall not attempt to outline the proof.

**Corollary 1.6.** *$H_1(B\Gamma_{g,b};\mathbb{Z}) = 0$ for all $b$ if $g \geq 4$.*

*Proof.* This follows easily from proposition 1.4 and theorem 1.5. □

By remark 1.3, there is a "universal" surface bundle $E \to B\Gamma_{g,b}$ with oriented fibers $\cong F_{g,b}$ and trivialized boundary bundle $\partial E \to B\Gamma_{g,b}$ (so that $\partial E$ is identified with $\partial F_{g,b}\times B\Gamma_{g,b}$). Let $T_vE$ be the vertical tangent bundle of $E$, a two-dimensional oriented vector bundle on $E$ with a trivialization over $\partial E$. This has an Euler class

$$e \in H^2(E, \partial E; \mathbb{Z}).$$

The image of $e^{i+1}$ under the Gysin transfer $H^{2i+2}(E, \partial E; \mathbb{Z}) \to H^{2i}(B\Gamma_{g,b}; \mathbb{Z})$ (alias integration along the fiber) is the Mumford–Morita–Miller characteristic class

$$\kappa_i \in H^{2i}(B\Gamma_{b,g}; \mathbb{Z}).$$

It was introduced by Mumford [26], but the description in differential topology language which we use here owes much to Miller [22] and Morita [24]. The class $\kappa_0$ equals the genus $g \in \mathbb{Z} \cong H^0(B\Gamma_{b,g}; \mathbb{Z})$. For $i > 0$, however, $\kappa_i$ is stable, i.e., independent of $g$ and $b$. Namely, the homomorphism $H^*(B\Gamma_{h,c}; \mathbb{Z}) \to H^*(B\Gamma_{g,b}; \mathbb{Z})$ induced by an embedding $F_{g,b} \to F_{h,c}$ as in theorem 1.5 takes the $\kappa_i$ class in $H^*(B\Gamma_{h,c}; \mathbb{Z})$ to the $\kappa_i$ class in $H^*(B\Gamma_{g,b}; \mathbb{Z})$. Mumford conjectured in [26] that the homomorphism of graded rings

$$\mathbb{Q}[x_1, x_2, x_3, \dots] \longrightarrow H^*(B\Gamma_{g,b}; \mathbb{Q})$$

taking $x_i$ to $\kappa_i$ (where $\deg(x_i) = 2i$) is an isomorphism in a (then unspecified) "stable range". By the Harer–Ivanov stability theorem, which is slightly younger than Mumford's conjecture, we can take that to mean: in degrees less than $g/2-1$. Morita [24], [25] and Miller [22] were able to show relatively quickly that Mumford's homomorphism $\mathbb{Q}[x_1, x_2, x_3, \dots] \longrightarrow H^*(B\Gamma_{g,b}; \mathbb{Q})$ is injective in the stable range. There matters stood until, in 1996-7, Tillmann introduced concepts from stable homotopy theory into surface bundle theory.

## 2. Stabilization and Tillmann's theorem

Here it will be convenient to consider oriented surfaces $F_{g,b}$ where each of the $b$ boundary circles is identified with $\mathbb{S}^1$. These identifications may or may not be orientation preserving; if it is, we regard the boundary component as "outgoing", otherwise as "incoming". We write $F_{g,b_1+b_2}$ to indicate that there are $b_1$ incoming and $b_2$ outgoing boundary circles.

Fix standard surfaces $F_{g,1+1}$ for $g \geq 0$ in such a way $F_{g+h,1+1}$ is identified with the union $F_{g,1+1} \sqcup_{\mathbb{S}^1} F_{h,1+1}$ (the outgoing boundary circle of $F_{g,1+1}$ being glued to the incoming boundary circle of $F_{h,1+1}$). A smooth automorphism $\alpha$ of $F_{g,1+1}$ , relative to the boundary, can be regarded as a smooth automorphism $\alpha \sqcup_{\mathbb{S}^1} \mathrm{id}$ of $F_{g,1+1} \sqcup_{\mathbb{S}^1} F_{1,1+1} \cong F_{g+1,1+1}$. This gives us stabilization homomorphisms

$$\cdots \longrightarrow \Gamma_{g,1+1} \longrightarrow \Gamma_{g+1,1+1} \longrightarrow \Gamma_{g+2,1+1} \longrightarrow \cdots$$

and we define $\Gamma_{\infty,1+1}$ as the direct limit $\mathrm{colim}_{g\to\infty}\Gamma_{g,1+1}$. This is the most obvious contender for the title of a stable mapping class group. It is still a perfect group. A more illuminating way to proceed is to note that a *pair* of smooth automorphisms $\alpha\colon F_{g,1+1} \to F_{g,1+1}$ and $\beta\colon F_{h,1+1} \to F_{h,1+1}$ , both relative to the boundary, determines an automorphism $\alpha \sqcup \beta$ of $F_{g+h,1+1}$. In other words, we have concatenation homomorphisms

$$\Gamma_{g,1+1} \times \Gamma_{h,1+1} \longrightarrow \Gamma_{g+h,1+1}$$

which induce maps $B\Gamma_{g,1+1} \times B\Gamma_{h,1+1} \to B\Gamma_{g+h,1+1}$. These maps amount to a structure of topological monoid on the disjoint union

$$\coprod_{g\geq 0} B\Gamma_{g,1+1}.$$

We can form the group completion $\Omega B(\coprod_g B\Gamma_{g,1+1})$. The inclusion of $\coprod_g B\Gamma_{g,1+1}$ in the group completion is a map of topological monoids and the target is a group–like topological monoid (i.e., its $\pi_0$ is a group) because it is a loop space.

**Proposition 2.1.** $\Omega B(\coprod_g B\Gamma_{g,1+1}) \simeq \mathbb{Z} \times B\Gamma^+_{\infty,1+1}$.

*Idea of proof.* It is enough to produce a map from right–hand side to left–hand side which induces an isomorphism in integer homology. Indeed, the existence of such a map implies that $H_1(\text{left–hand side}; \mathbb{Z}) = 0$. Since the left–hand side is a loop space, the vanishing of $H_1$ implies that all its connected components are simply connected.
Let $\mathcal{M} = \coprod_g B\Gamma_{g,1+1}$ and let $\mathcal{F}$ be the homotopy direct limit (here: telescope) of the sequence

$$\mathcal{M} \xrightarrow{z\cdot} \mathcal{M} \xrightarrow{z\cdot} \mathcal{M} \xrightarrow{z\cdot} \mathcal{M} \xrightarrow{z\cdot} \cdots$$

where $z\cdot$ is left multiplication by a fixed element in the genus one component of $\mathcal{M}$. The topological monoid $\mathcal{M}$ acts on the right of $\mathcal{F}$. Theorem 1.5 implies that it acts by maps $\mathcal{F} \to \mathcal{F}$ which induce isomorphisms in integer homology. It follows [21] that the projection from the Borel construction $\mathcal{F}_{h\mathcal{M}}$ to the classifying space $B\mathcal{M}$ is a *homology fibration*. (The Borel construction $\mathcal{F}_{h\mathcal{M}}$ is the classifying space of the topological category with object space $\mathcal{F}$ and morphism space $\mathcal{F} \times \mathcal{M}$, where the "source" map is the projection $\mathcal{F} \times \mathcal{M} \to \mathcal{F}$, the "target" map is the right action map $\mathcal{F} \times \mathcal{M} \to \mathcal{F}$, and composition of morphisms is determined by the multiplication in $\mathcal{M}$.) In particular, the inclusion of the fiber of

$$\mathcal{F}_{h\mathcal{M}} \longrightarrow B\mathcal{M}$$

over the base point into the corresponding homotopy fiber induces an isomorphism in integer homology. Since the fiber over the base point is $\mathcal{F} \simeq \mathbb{Z} \times B\Gamma_{\infty,1+1}$, it remains only to identify the *homotopy* fiber over the base point as $\Omega B(\mathcal{M})$. For that it is enough to show that $\mathcal{F}_{h\mathcal{M}}$ is contractible. But $\mathcal{F}_{h\mathcal{M}}$ is the homotopy direct limit (telescope) of the sequence

$$\mathcal{M}_{h\mathcal{M}} \xrightarrow{\cdot z} \mathcal{M}_{h\mathcal{M}} \xrightarrow{\cdot z} \mathcal{M}_{h\mathcal{M}} \xrightarrow{\cdot z} \mathcal{M}_{h\mathcal{M}} \xrightarrow{\cdot z} \cdots$$

where each term $\mathcal{M}_{h\mathcal{M}}$ is contractible. □

One remarkable consequence of proposition 2.1 is that $\mathbb{Z} \times B\Gamma^+_{\infty,1+1}$ is a loop space. Miller did better than that [22] by constructing a two–fold loop space structure on $\mathbb{Z} \times B\Gamma^+_{\infty,1+1}$. To be more accurate, he constructed such a structure on a space which ought to be denoted

$$\mathbb{Z} \times B\Gamma^+_{\infty,0+1}$$

but which is homotopy equivalent to $\mathbb{Z} \times B\Gamma^+_{\infty,1+1}$ by the Harer–Ivanov theorem. This construction of Miller's will not be explained here (perhaps unfairly, because it may have influenced the proof of the following theorem due to Tillmann).

**Theorem 2.2.** [34] *The space $\mathbb{Z} \times B\Gamma^+_{\infty,1+1}$ is an infinite loop space.*

*Remark.* If $Y$ is an infinite loop space, then the contravariant functor taking a space $X$ to $[X, Y]$, the set of homotopy classes of maps from $X$ to $Y$, is the 0–th term of a generalized cohomology theory. Apart from Eilenberg–MacLane spaces, the most popular example is $Y = \mathbb{Z} \times B\mathrm{U}$, which is an infinite loop space because it is homotopy equivalent to its own two–fold loop space. The corresponding generalized cohomology theory is, of course, the $K$–theory of Atiyah, Bott and Hirzebruch. The construction, description, classification, etc., of generalized cohomology theories is considered to be a major part of stable homotopy theory.

*Outline of proof of theorem 2.2.* It is well known that infinite loop spaces can be manufactured from symmetric monoidal categories, i.e., categories with a notion of "direct sum" which is associative and commutative up to canonical isomorphisms. For more details on symmetric monoidal categories, see [1]. If $\mathcal{C}$ is such a category, then the classifying space $B\mathcal{C}$ has a structure of topological monoid which reflects the direct sum operation in $\mathcal{C}$. If this happens to be group–like, i.e., $\pi_0 B\mathcal{C}$ is a group, then $B\mathcal{C}$ is an infinite loop space. If not, then at least the group completion $\Omega B(B\mathcal{C})$ is an infinite loop space. More details and a particularly satisfying proof can be found in [32]. For an overview and alternative proofs, see also [1].
The standard example of such a category is the category of finitely generated left projective modules over a ring $R$, where the morphisms are the $R$–isomorphisms. Here group completion of the classifying space is required and the resulting infinite loop space is the algebraic $K$–theory space $K(R)$. For a slightly different example, take the category of finite dimensional vector spaces over $\mathbb{C}$, with $\mathrm{mor}(V, W)$ equal to the *space* of $\mathbb{C}$–linear isomorphisms from $V$ to $W$. The new feature here is that we have a symmetric monoidal category with a topology on each of its morphism sets. This "enrichment" must be fed into the construction of the classifying space, which then turns out to be homotopy equivalent to $\coprod_n B\mathrm{U}(n)$. Again, group completion is required and the associated infinite loop space is $\mathbb{Z} \times B\mathrm{U}$, up to a homotopy equivalence.
Another example which is particularly important here is as follows. Let $\mathrm{ob}(\mathcal{C})$ consist of all closed oriented 1–manifolds. Given two such objects, say $C$ and $C'$, we would like to say roughly that a morphism from $C$ to $C'$ is a smooth compact surface $F$ with boundary $-C \sqcup C'$ (where the minus sign indicates a reversed orientation). To be more precise, let $\mathrm{mor}(C, C')$ be "the" classifying space for bundles of smooth compact oriented surfaces whose boundaries are identified with the disjoint union $-C \sqcup C'$. The composition map

$$\mathrm{mor}(C, C') \times \mathrm{mor}(C', C'') \longrightarrow \mathrm{mor}(C, C'')$$

is given by concatenation, as usual. Disjoint union of objects and morphisms can be regarded as a "direct sum" operation which makes $\mathcal{C}$ into a symmetric monoidal category, again with a topology on each of its morphism sets. The enrichment must be fed into the construction of $B\mathcal{C}$. Then $B\mathcal{C}$ is clearly connected, and by the above it is an infinite loop space.
Unfortunately it is not clear whether the homotopy type of $B\mathcal{C}$ is at all closely related to that of $\mathbb{Z} \times B\Gamma^+_{\infty,1+1}$. This is mostly due to the fact that, in the above definition of $\mathrm{mor}(C, C')$ for objects $C$ and $C'$ of $\mathcal{C}$, we allowed arbitrary compact surfaces with boundary $\cong -C \sqcup C'$ instead of insisting on *connected* surfaces. And if we had insisted on connected surfaces throughout, we would have lost the "direct sum" alias "disjoint union" operation which is so essential. (Disjoint unions of

connected things are typically not connected.) A new idea is required, and Tillmann comes up with the following beautiful two–liner.
Make a subcategory $\mathcal{C}_0$ of $\mathcal{C}$ by keeping all objects of $\mathcal{C}$, but only those morphisms (surfaces) for which the inclusion of the outgoing boundary induces a surjection in $\pi_0$. In the above notation, where we have a surface $F$ and $\partial F$ is identified with $-C \sqcup C'$, the condition means that $\pi_0 C' \to \pi_0 F$ is onto.
It is clear that $\mathcal{C}_0$ is closed under the disjoint union operation, and that $B\mathcal{C}_0$ is connected, so $B\mathcal{C}_0$ is still an infinite loop space. While the surfaces which we see in the definition of $\mathcal{C}_0$ need not be connected, they always become connected when we compose on the left (i.e., concatenate at the outgoing boundary) with a morphism to the connected object $\mathbb{S}^1$. This observation leads fairly automatically, i.e., by imitation of the proof of proposition 2.1, to a homotopy equivalence

$$\Omega(B\mathcal{C}_0) \simeq \mathbb{Z} \times B\Gamma^+_{\infty,1+1}$$

and so to the conclusion that $\mathbb{Z} \times B\Gamma^+_{\infty,1+1}$ is an infinite loop space.
Namely, we introduce a contravariant functor $\mathcal{F}$ on $\mathcal{C}_0$ in such a way that $\mathcal{F}(C)$, for an object $C$, is the homotopy direct limit (= telescope) of the sequence

$$\mathrm{mor}_{\mathcal{C}_0}(C,\mathbb{S}^1) \xrightarrow{z\cdot} \mathrm{mor}_{\mathcal{C}_0}(C,\mathbb{S}^1) \xrightarrow{z\cdot} \mathrm{mor}_{\mathcal{C}_0}(C,\mathbb{S}^1) \xrightarrow{z\cdot} \cdots$$

where $z\cdot$ is left multiplication by a fixed element in the genus one component of $\mathrm{mor}_{\mathcal{C}_0}(\mathbb{S}^1,\mathbb{S}^1)$. Theorem 1.5 implies that any map $\mathcal{F}(C') \to \mathcal{F}(C)$ determined by a morphism $C \to C'$ in $\mathcal{C}_0$ induces an isomorphism in integer homology. It follows that the projection from the homotopy direct limit of $\mathcal{F}$ to $B\mathcal{C}_0$ is a homology fibration. (The homotopy colimit of $\mathcal{F}$ replaces the Borel construction in the proof of proposition 2.1; see definition 6.1 below for more details.) The fiber over the vertex determined by the object $\mathbb{S}^1$ is

$$\mathcal{F}(\mathbb{S}^1) \simeq \mathbb{Z} \times B\Gamma_{\infty,1+1}\,.$$

It remains to show that the corresponding homotopy fiber is $\simeq \Omega B\mathcal{C}_0$, and for that it is enough to prove that $\mathrm{hocolim}\,\mathcal{F}$ is contractible. But $\mathrm{hocolim}\,\mathcal{F}$ is the homotopy direct limit (telescope) of a sequence

$$\mathrm{hocolim}\,\mathcal{E} \xrightarrow{z\cdot} \mathrm{hocolim}\,\mathcal{E} \xrightarrow{z\cdot} \mathrm{hocolim}\,\mathcal{E} \xrightarrow{z\cdot} \mathrm{hocolim}\,\mathcal{E} \xrightarrow{z\cdot} \cdots$$

where $\mathcal{E}$ is the representable contravariant functor $C \mapsto \mathrm{mor}_{\mathcal{C}_0}(C,\mathbb{S}^1)$. Homotopy colimits of representable contravariant functors (on categories where the morphism sets are topologized and composition of morphisms is continuous) are always contractible. □

*Remark.* The outline above is deliberately careless about the definition of the composition maps (alias concatenation maps)

$$\mathrm{mor}(C,C') \times \mathrm{mor}(C',C'') \longrightarrow \mathrm{mor}(C,C'')$$

in the category $\mathcal{C}$. This is actually not a straightforward matter. Tillmann has a very elegant solution in a later article [35] where she constructs a category equivalent to the $\mathcal{C}_0$ above using (few) generators and relations.

## 3. Mock surface bundles

Relying on theorem 2.2, Tillmann in [35] began to develop methods to split off known infinite loop spaces from $\mathbb{Z}\times B\Gamma^+_{\infty,1+1}$ , specifically infinite loop spaces of the “free” type

$$Q(X) = \operatorname*{colim}_{n\to\infty} \Omega^n\Sigma^n X$$

where $X$ is a pointed space. This was taken to a higher level in a joint paper by Madsen and Tillmann [19]. The paper begins with the construction of an integral version of the total Mumford–Morita–Miller class, which is an infinite loop map $\alpha_\infty$ from $\mathbb{Z}\times B\Gamma^+_{\infty,1+1}$ to a well known infinite loop space $\Omega^\infty\mathbb{C}\mathbf{P}^\infty_{-1}$. The main result is a splitting theorem, formulated in terms of $\alpha_\infty$ and known decompositions of $\Omega^\infty\mathbb{C}\mathbf{P}^\infty_{-1}$, which can be regarded as a $p$–local version of the Morita–Miller injectivity result. It is proved by methods which are somewhat similar to Morita's methods. Here we are going to describe $\alpha_\infty$ from a slightly different angle, emphasizing bordism theoretic ideas and initially downplaying the motivations from characteristic class theory.

**Definition 3.1.** Let $X$ be a smooth manifold (with empty boundary). A *mock surface bundle* on $X$ consists of a smooth manifold $M$ with $\dim(M)-\dim(X)=2$, a proper smooth map

$$q\colon M\to X\,,$$

a stable vector bundle surjection $\delta q\colon TM\to q^*TX$ and an orientation of the two–dimensional kernel vector bundle $\ker(\delta q)$ on $M$.

*Explanations.* The word *stable* in “stable vector bundle surjection” means that $\delta p$ is a vector bundle map $TM\times\mathbb{R}^i\to p^*TX\times\mathbb{R}^i$ for some $i$, possibly large. Note that $\delta q$ is not required to agree with the differential $dq$ of $q$. It should be regarded as a “formal” differential of $q$. If $\delta q = dq$, then $q$ is a smooth proper submersion. Smooth proper submersions are fiber bundles by Ehresmann's lemma [5]. In short, an *integrable* mock surface bundle ($\delta q = dq$) is a surface bundle.

Mock surface bundles share many good properties with honest surface bundles. They can (usually) be pulled back, they have a classifying space, and they have Mumford–Morita–Miller characteristic classes, as we shall see.
To begin with the pullback property, suppose that $q\colon M\to X_2$ with $\delta q$ etc. is a mock surface bundle and let $f\colon X_1\to X_2$ be a smooth map. If $f$ is transverse to $q$, which means that the map $(x,y)\mapsto(f(x),q(y))$ from $X_1\times M$ to $X_2\times X_2$ is transverse to the diagonal, then the pullback

$$f^*M=\{(x,y)\in X_1\times M\mid f(x)=q(y)\}$$

is a smooth manifold, with projection $p\colon f^*M\to X_1$. The transversality property and the information in $\delta q$ can be used to make a canonical choice of formal (stable) differential

$$\delta p\colon T(f^*M)\longrightarrow p^*TX_1$$

with oriented two–dimensional kernel bundle. Then $(p,\delta p)$ is a mock surface bundle on $X_1$. The details are left to the reader. If $f$ is not transverse to $q$, then we can make it transverse to $q$ by a small perturbation [5, 14.9.3]. In that situation, of course, $(p,\delta p)$ is not entirely well defined because it depends on the perturbation. It is however well defined up to a *concordance*:

**Definition 3.2.** Two mock surface bundles $q_0 : M_0 \to X$ and $q_1 : M_1 \to X$ (with vector bundle data which we suppress) are *concordant* if there exists a mock surface bundle $q_{\mathbb{R}} : M_{\mathbb{R}} \to X \times \mathbb{R}$ (with vector bundle data ...) such that $q_{\mathbb{R}}$ is transverse to $X \times \{0\}$ and $X \times \{1\}$, and the pullbacks of $q_{\mathbb{R}}$ to $X \times \{0\}$ and $X \times \{1\}$ agree with $q_0 \times \{0\}$ and $q_1 \times \{1\}$, respectively.

Next we turn to the construction of a classifying space for mock surface bundles. This is an instance of Pontryagin–Thom theory in a cohomological setting which was popularized by Quillen [9] and later by Buoncristiano–Rourke–Sanderson [7]. Let $\mathrm{Gr}_2(\mathbb{R}^{2+n})$ be the Grassmannian of oriented 2-planes in $\mathbb{R}^{2+n}$ and let $P_n$, $V_n$ be the canonical vector bundles of dimension 2 and $n$ on $\mathrm{Gr}_2(\mathbb{R}^{2+n})$, respectively. Let $\mathrm{Th}(V_n)$ be the Thom space (one–point compactification of the total space) of $V_n$. Since $V_{n+1}|\mathrm{Gr}_2(\mathbb{R}^{2+n})$ is identified with $V_n \times \mathbb{R}$, there is a preferred embedding $\Sigma\mathrm{Th}(V_n) \to \mathrm{Th}(V_{n+1})$, with adjoint $\mathrm{Th}(V_n) \to \Omega\mathrm{Th}(V_{n+1})$. We form the direct limit

$$\operatorname*{colim}_{n\to\infty} \ \Omega^{n+2}\mathrm{Th}(V_n) \ =: \Omega^\infty\mathbb{C}\mathbf{P}^\infty_{-1}\,.$$

**Lemma 3.3.** *For any smooth manifold $X$ there is a natural bijection from the set of homotopy classes $[X, \Omega^\infty\mathbb{C}\mathbf{P}^\infty_{-1}]$ to the set of concordance classes of mock surface bundles on $X$.*

*Outline of proof (one direction only).* A map from $X$ to $\Omega^\infty\mathbb{C}\mathbf{P}^\infty_{-1}$ factors through $\Omega^{n+2}\mathrm{Th}(V_n)$ for some $n$. Let $f$ be the adjoint, a based map from the $(n+2)$–fold suspension of $X_+$ to $\mathrm{Th}(V_n)$. It is convenient to identify the complement of the base point in $\Sigma^{n+2}X_+$ with $X \times \mathbb{R}^{n+2}$. We can assume that $f$ is transverse to the zero section of $V_n$. Let

$$M \subset X \times \mathbb{R}^{n+2}$$

be the inverse image of the zero section under $f$. Let $q : M \to X$ be the projection. By construction of $M$ there is an isomorphism

$$TM \oplus (f|M)^*V_n \ \cong\ q^*TX \times \mathbb{R}^{n+2}$$

of vector bundles on $M$. Adding $(f|M)^*P_n$ on the left hand side and noting that $TM \oplus (f|M)^*V_n \oplus (f|M)^*P_n$ is identified with $TM \times \mathbb{R}^{n+2}$, we get a vector bundle surjection

$$\delta q : TM \times \mathbb{R}^{n+2} \longrightarrow q^*TX \times \mathbb{R}^{n+2}$$

with $\ker(\delta q) \cong (f|M)^*P_n$ , which implies an orientation on $\ker(\delta q)$. Now $q : M \to X$ and $\delta q$ with the orientation on $\ker(\delta q)$ constitute a mock surface bundle whose concordance class is independent of all the choices we made in the construction. □

Finally we construct Mumford–Morita–Miller classes for mock surface bundles. Let $q : M \to X$ be a mock surface bundle, with $\delta q : TM \to q^*TX$. The oriented 2–dimensional vector bundle $\ker(\delta q)$ on $M$ has an Euler class $e \in H^2(M; \mathbb{Z})$. Our hypotheses on $q$ imply that $q$ induces a transfer map in cohomology,

$$H^{*+2}(M; \mathbb{Z}) \longrightarrow H^*(X; \mathbb{Z}).$$

This is obtained essentially by conjugating an induced map in homology with Poincaré duality. (The correct version of homology for this purpose is *locally finite* homology with $\mathbb{Z}$–coefficients twisted by the orientation character.) We now define $\kappa_i(q, \delta q) \in H^{2i}(X; \mathbb{Z})$ to be the image of $e^{i+1} \in H^{2i+2}(M; \mathbb{Z})$ under the transfer. The classes $\kappa_i$ are concordance invariants and behave naturally under (transverse) pullback of mock surface bundles. They can therefore be regarded as classes in the cohomology of the classifying space for mock surface bundles:

$$\kappa_i \in H^{2i}(\Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}; \mathbb{Z})\,.$$

It is not difficult to see that certain mild modifications of definition 3.1 do not change the concordance classification. In particular, a convenient modification of that sort consists in allowing $q\colon M \to X$ with $\delta q$ etc. where $M$ has a boundary $\partial M$, the restriction $q|\partial M$ is a trivialized bundle with fibers $\cong -\mathbb{S}^1 \sqcup \mathbb{S}^1$ and $\delta q$ agrees with the differential $dq$ on $\partial M$. If we now regard $\Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}$ as a classifying space for these modified mock surface bundles, then we obtain a comparison map of classifying spaces

$$\coprod_g B\Gamma_{g,1+1} \longrightarrow \Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}$$

(Indeed, the left–hand side is a classifying space for honest bundles whose fibers are *connected* oriented smooth surfaces with prescribed boundary $\cong -\mathbb{S}^1 \sqcup \mathbb{S}^1$.) Furthermore, the map commutes with concatenation and its target is a group–like space. By the universal property of the group completion, the map just constructed extends in an essentially unique way to a map

$$\alpha_\infty\colon \mathbb{Z} \times B\Gamma^+_{\infty,1+1} \longrightarrow \Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}$$

(where we are using proposition 2.1). One of us (I.M.) conjectured the following, now a theorem [20]:

**Theorem 3.4.** *The map $\alpha_\infty$ is a homotopy equivalence.*

As a conjecture this is stated in [19], and supported by the splitting theorem mentioned earlier. In the same article, it is shown that $\alpha_\infty$ is a map of infinite loop spaces, with Tillmann's infinite loop space structure on $\mathbb{Z} \times B\Gamma^+_{\infty,1+1}$, and the obvious infinite loop structure on

$$\Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1} = \operatorname*{colim}_{n\to\infty} \Omega^{n+2}\mathrm{Th}(V_n).$$

It is easy to show that the rational cohomology of any connected component of $\Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}$ is a polynomial ring $\mathbb{Q}[x_1, x_2, x_3, \dots]$ where $\deg(x_i) = 2i$; moreover the $x_i$ can be taken as the $\kappa_i$ classes for $i > 0$. The cohomology with finite field coefficients

$$H^*(\Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}; \mathbb{F}_p)$$

is much more difficult to determine. Nevertheless this has been done in the meantime by Galatius [13].

*Remark on notation.* The strange abbreviation $\Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}$ for the direct limit $\mathrm{colim}_n\, \Omega^{n+2}\mathrm{Th}(V_n)$ can be justified as follows. Let $\mathbb{C}P^n \subset \mathrm{Gr}_2(\mathbb{R}^{2n+2})$ be the Grassmannian of one–dimensional $\mathbb{C}$–linear subspaces in $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$, alias complex projective space of complex dimension $n$. Let $L_n$ be the tautological line bundle on $\mathbb{C}P^n$ and $L_n^\perp$ its canonical complement, a complex vector bundle of dimension $n$. The inclusion

$$\operatorname*{colim}_{n\to\infty} \Omega^{2n+2}\mathrm{Th}(L_n^\perp) \longrightarrow \operatorname*{colim}_{2n\to\infty} \Omega^{2n+2}\mathrm{Th}(V_{2n})$$

is a homotopy equivalence. Now Thom spaces of certain vector bundles on (complex) projective spaces can be viewed as "stunted" projective spaces

$$\mathbb{C}P^i_k = \mathbb{C}P^i/\mathbb{C}P^{k-1}$$

where $i \geq k$. Namely, $\mathbb{C}P^i_k$ is identified with the Thom space of the Whitney sum of $k$ copies of the tautological line bundle on $\mathbb{C}P^{i-k}$. Allowing $k = -1$, stable homotopy theorists therefore like to write

$$\mathrm{Th}(L_n^\perp) = \Sigma^{2n+2}\mathrm{Th}(-L_n) = \Sigma^{2n+2}\mathbb{C}P^{n-1}_{-1}.$$

In addition they use the reasonable abbreviation

$$\operatorname*{colim}_{n\to\infty}\ \Omega^{n+2}\Sigma^{2n+2}\mathbb{C}P^{n-1}_{-1} =: \Omega^\infty\mathbb{C}\mathbf{P}^\infty_{-1}.$$

## 4. First desingularization

In the remaining sections, some key ideas from the proof of theorem 3.4 in [20] will be sketched. The proof proceeds from the target of $\alpha_\infty$ to the source. That is, it starts from the original (co)bordism–theoretic description of $\Omega^\infty\mathbb{C}\mathbf{P}^\infty_{-1}$ and goes through a number of steps to obtain alternative descriptions which are more and more bundle theoretic. Each step can also be viewed as a step towards the goal of "desingularizing" mock surface bundles. The first step in this sequence is a little surprising.

Let $q\colon M \to X$ together with $\delta q\colon TM \to q^*TX$ be a mock surface bundle. We form $E = M\times\mathbb{R}$ and get $(p,f)\colon E \to X\times\mathbb{R}$ where $p(z,t) = q(z)$ and $f(z,t) = t$ for $(z,t)\in M\times\mathbb{R} = E$. There is a formal (surjective, stable) differential

$$\delta p\colon TE \to p^*TX\,,$$

obtained by composing the projection $TE \to TM$ with $\delta q$. There is also the honest differential of $f$, which we regard as a vector bundle surjection

$$\delta f = df\colon \ker(\delta p) \to f^*(T\mathbb{R}).$$

All in all, we have made a conversion

$$(q,\delta q) \rightsquigarrow (p,f,\delta p,\delta f).$$

Here $(p,f)\colon E \to X\times\mathbb{R}$ is smooth and proper, $\delta p$ is a formal (stable, surjective) differential for $p$ with a 3–dimensional oriented kernel bundle, and $\delta f$ is a surjective vector bundle map from $\ker(\delta p)$ to $f^*(T\mathbb{R})$ (which agrees with $df$).
We are going to "sacrifice" the equation $\delta f = df$ in order to "obtain" an equation $\delta p = dp$. It turns out that this can always be achieved by a continuous deformation

$$((p_s,f_s,\delta p_s,\delta f_s))_{s\in[0,1]}$$

of the quadruple $(p,f,\delta p,\delta f)$, on the understanding that each $(p_s,f_s)\colon E\to X\times\mathbb{R}$ is smooth and proper, each $\delta p_s$ is a formal (stable, surjective) differential for $p_s$ with a 3–dimensional oriented kernel bundle, and each $\delta f_s\colon \ker(\delta p_s) \to f_s^*(T\mathbb{R})$ is a surjective vector bundle map. (For $s = 0$ we want $(p_s,f_s,\delta p_s,\delta f_s) = (p,f,\delta p,\delta f)$ and for $s = 1$ we want $\delta p_s = dp_s$, so that $p_1$ is a submersion.)

The proof is easy modulo submersion theory [28], [14], especially if $X$ is closed which we assume for simplicity. Firstly, obstruction theory [33] shows that $\delta p$ , although assumed to be a *stable* vector bundle surjection, can be deformed (through stable vector bundle surjections) to an honest vector bundle surjection $\delta_u p$ from $TE$

to $p^*TX$. Secondly, the manifold $E$ has no compact component, so that the main theorem of submersion theory applies to $E$ and the pair $(p, \delta_u p)$. The combined conclusion is that $(p, \delta p)$ can be deformed through similar pairs $(p_s, \delta p_s)$ to an *integrable* pair $(p_1, \delta p_1)$, so that $\delta p_1 = dp_1$ and consequently $p_1$ is a submersion. We set $f_s = f$ for all $s \in [0,1]$. Finally, since $\ker(\delta p_s) \cong \ker(\delta p)$ for each $s$, there is no problem in defining $\delta f_s \colon \ker(\delta p_s) \to f_s^*(T\mathbb{R})$ somehow, for all $s \in [0,1]$ as a surjective vector bundle map depending continuously on $s$. Note that the maps $(p_s, f_s)\colon E \to X \times \mathbb{R}$ are automatically proper since each $f_s = f$ is proper.

These observations amount to an outline of more than half the proof of the following proposition.

**Proposition 4.1.** *The classifying space for mock surface bundles, $\Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}$ , is also a classifying space for families of oriented smooth 3–manifolds $E_x$ equipped with a proper map $f_x\colon E_x \to \mathbb{R}$ and a vector bundle surjection $\delta f_x \colon TE_x \to f_x^*(T\mathbb{R})$.*

*Details.* The "families" in proposition 4.1 are submersions $\pi\colon E \to X$ with fibers $E_x$ for $x \in X$. They are *not* assumed to be bundles. The parameter space $X$ can be any smooth manifold without boundary (and in some situations it is convenient to allow a nonempty boundary). The maps $f_x\colon E_x \to \mathbb{R}$ are supposed to make up a smooth map $f\colon E \to \mathbb{R}$. Similarly the $\delta f_x$ make up a vector bundle surjection $\delta f$ from the vertical tangent bundle of $E$ to $f^*(T\mathbb{R})$. The properness condition, correctly stated, means that $(\pi, f)\colon E \to X \times \mathbb{R}$ is proper.
Although these families are submersions rather than bundles, they can be pulled back just like bundles. The classification is up to concordance. A concordance between two families on $X$ (of the sort under discussion) is another family (of the sort under discussion) on $X \times \mathbb{R}$, restricting to the prescribed families on the submanifolds $X \times \{0\}$ and $X \times \{1\}$.

*Outline of remainder of proof of proposition 4.1.* We have seen how a mock surface bundle on $X$ can be converted to a family as in proposition 4.1. Going in the other direction is easier: namely, given a family $\pi\colon E \to X$ with $f\colon E \to \mathbb{R}$ etc., as in proposition 4.1, choose a regular value $c \in \mathbb{R}$ for $f$ and let $M = f^{-1}(c) \subset E$. Then $q = \pi|M$ etc. is a mock surface bundle on $X$.
In showing that these two procedures are inverses of one another, we have to verify in particular the following. Given a family $\pi\colon E \to X$ with $f\colon E \to \mathbb{R}$ etc., as in proposition 4.1, and a regular value $c \in \mathbb{R}$ for $f$ with $M = f^{-1}(c)$, there exists a concordance from the original family to another family with total space $\cong M \times \mathbb{R}$. This is particularly easy to see when $X$ is compact (i.e., closed). In that case we can choose a small open interval $U$ about $c \in \mathbb{R}$ containing no critical values of $f$, and an orientation preserving diffeomorphism $h\colon U \to \mathbb{R}$. Let $E' = f^{-1}(U) \cong M \times \mathbb{R}$. Now $\pi|E'$ together with $h \circ f|E'$ and $dh \circ \delta f$ constitute a new family which is concordant to the old one. (To make the concordance, use an isotopy from $\mathrm{id}\colon \mathbb{R} \to \mathbb{R}$ to $h^{-1}$.) Yes, the concordance relation is very coarse. □

## 5. A zoo of generalized surfaces

The advantage of the new characterization of $\Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}$ given in proposition 4.1 is that it paves the way for a number of useful variations on the Madsen conjecture alias theorem 3.4. We are going to formulate these as statements about classifying spaces for families of certain generalized ("thickened") surfaces. Following is a list of the types of generalized or thickened surface which we need, with labels. (They are all defined as 3–manifolds with additional structure; but see the comments below.)

| | |
|---|---|
| $\mathcal{V}$ | oriented smooth 3–manifold $E_x$ with proper smooth nonsingular $f_x\colon E_x \to \mathbb{R}$ |
| $\mathcal{W}$ | oriented smooth 3–manifold $E_x$ with proper smooth Morse function $f_x\colon E_x \to \mathbb{R}$ |
| $\mathcal{W}_{\rm loc}$ | oriented smooth 3–manifold $E_x$ with smooth Morse function $f_x\colon E_x \to \mathbb{R}$ whose restriction to the critical point set $\mathrm{crit}(f_x)$ is proper |
| $h\mathcal{V}$ | oriented smooth 3–manifold $E_x$ with proper $f_x\colon E_x \to \mathbb{R}$ and vector bundle surjection $\delta f_x\colon TE_x \to f_x^*(T\mathbb{R})$ |
| $h\mathcal{W}$ | oriented smooth 3–manifold $E_x$ with proper $f_x\colon E_x \to \mathbb{R}$ and $\delta f_x\colon TE_x \to f_x^*(T\mathbb{R})$ of Morse type (details below) |
| $h\mathcal{W}_{\rm loc}$ | oriented smooth 3–manifold $E_x$ with $f_x\colon E_x \to \mathbb{R}$ and $\delta f_x\colon TE_x \to f_x^*(T\mathbb{R})$ of Morse type; restriction of $f_x$ to $\mathrm{crit}(\delta f_x)$ is proper (details below) |

*Details.* The map $\delta f$ of "Morse type" in the definition of types $h\mathcal{W}$ and $h\mathcal{W}_{\rm loc}$ is a map $TE_x \to f_x^*(T\mathbb{R})$ over $E$, but is not required to be a vector bundle homomorphism. It is required to be the sum of a linear term $\ell$ and a quadratic term $k$, subject to the condition that $k_z$ is nondegenerate whenever $\ell_z = 0$, for $z \in E_x$. Its formal critical point set $\mathrm{crit}(\delta f_x)$ is the the set of $z \in E_x$ such that $\ell_z = 0$.

*Comments.* The conditions "proper" and "nonsingular" in the definition of type $\mathcal{V}$ imply that $f_x\colon E_x \to \mathbb{R}$ is a proper submersion, hence a bundle of closed surfaces on $\mathbb{R}$. From a classification point of view, this carries the same information as the closed surface $f_x^{-1}(0)$. Similarly, in the definitions of type $\mathcal{W}$ and $\mathcal{W}_{\rm loc}$, the focus is mainly on $f_x^{-1}(0)$, which in both cases is a surface with finitely many very "moderate" singularities. (It is compact in the $\mathcal{W}$ case, but can be noncompact in the $\mathcal{W}_{\rm loc}$ case).
The $x$ superscripts have been kept mainly for consistency with the formulation of proposition 4.1. They do indicate, correctly, that we are interested in families of such generalized surfaces.

Let $|\mathcal{V}|$, $|\mathcal{W}|$, $|\mathcal{W}_{\rm loc}|$, $|h\mathcal{V}|$, $|h\mathcal{W}|$ and $|h\mathcal{W}_{\rm loc}|$ be the classifying spaces for families of generalized surfaces of type $\mathcal{V}$, $\mathcal{W}$, $\mathcal{W}_{\rm loc}$, $h\mathcal{V}$, $h\mathcal{W}$ and $h\mathcal{W}_{\rm loc}$, respectively. We have seen the details in the case of $h\mathcal{V}$; they are similar in the other cases. In particular, *family with parameter manifold* $X$ should always be interpreted as *submersion with target* $X$. (The existence of the six classifying spaces can be deduced from a general statement known as Brown's representation theorem [6], but more explicit constructions are available. In the $\mathcal{V}$ case, the families alias submersions are automatically bundles with fibers $E_x \cong F_x \times \mathbb{R}$, where $F_x$ is a closed surface.) We obtain a commutative diagram of classifying spaces

$$\begin{array}{ccccc} |\mathcal{V}| & \longrightarrow & |\mathcal{W}| & \longrightarrow & |\mathcal{W}_{\rm loc}| \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ |h\mathcal{V}| & \longrightarrow & |h\mathcal{W}| & \longrightarrow & |h\mathcal{W}_{\rm loc}| \end{array} \qquad (*)$$

where the vertical arrows are obtained essentially by viewing honest derivatives as "formal" derivatives. One of the six spaces, $|\mathcal{V}|$, is a little provisional because it classifies *all* bundles of closed surfaces (whereas we should be interested in connected surfaces of high genus). The other five, however, are in final form. We saw that $|h\mathcal{V}| \simeq \Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}$. Modulo a plus construction and small corrections in the definition of $|\mathcal{V}|$, the left–hand vertical arrow in the diagram is $\alpha_\infty$.

**Proposition 5.1.** *The lower row of diagram* $(*)$ *is a homotopy fiber sequence.*

**Lemma 5.2.** *The right–hand vertical arrow in* $(*)$ *is a homotopy equivalence.*

*About the proofs.* The proof of proposition 5.1 is a matter of stable homotopy theory and specifically bordism theory. The spaces $|h\mathcal{W}|$ and $|h\mathcal{W}_{\rm loc}|$ have alternative bordism–theoretic descriptions similar to the equivalence

$$|h\mathcal{V}| \simeq \operatorname*{colim}_{n\to\infty} \ \Omega^{n+2}\mathrm{Th}(V_n)$$

of proposition 4.1. In particular, let $\mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{n+3})$ be the Grassmannian of 3–dimensional oriented linear subspaces of $\mathbb{R}^{n+3}$ equipped with a function $\ell + k$ of Morse type (where $\ell$ is a linear form and $k$ is a quadratic form). Let $W_n$ be the canonical $n$–dimensional vector bundle on $\mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{n+3})$. Then

$$|h\mathcal{W}| \simeq \operatorname*{colim}_{n\to\infty} \ \Omega^{n+2}\mathrm{Th}(W_n)\,.$$

From the bordism–theoretic descriptions, it follows easily that the lower row of $(*)$ is a homotopy fiber sequence.
The proof of lemma 5.2 is easy. Apart from the fact that $|h\mathcal{W}_{\rm loc}|$ is well understood in bordism–theoretic terms, the main reason for that is as follows: A generalized surface $(E_x, f_x)$ of of type $\mathcal{W}_{\rm loc}$ is determined, up to a canonical concordance, by its germ about the critical point set of $f_x$. This carries over to families of surfaces of type $\mathcal{W}_{\rm loc}$.

**Theorem 5.3.** *The middle vertical arrow in* $(*)$ *is a homotopy equivalence.*

This is a distant corollary of a hard theorem due to Vassiliev [36], [37]. Following are some definitions and abbreviations which are useful in the formulation of Vassiliev's theorem.
Let $M$ be a smooth manifold without boundary, $z \in M$. A *k–jet* from $M$ to $\mathbb{R}^n$ at $z$ is an equivalence class of smooth map germs $f\colon (M,z) \to \mathbb{R}^n$, where two such germs are considered equivalent if they agree to $k$–th order at $z$. Let $J^k(M,\mathbb{R}^n)_z$ be the set of equivalence classes and let

$$J^k(M,\mathbb{R}^n) = \bigcup_z J^k(M,\mathbb{R}^n)_z.$$

The projection $J^k(M,\mathbb{R}^n) \to M$ has a canonical structure of smooth vector bundle. Every smooth function $f\colon M \to \mathbb{R}^n$ determines a smooth section $j^k f$ of the jet bundle $J^k(M,\mathbb{R}^n) \to M$, the *k–jet prolongation* of $f$. The value of $j^k f$ at $z \in M$ is the $k$–jet of $f$ at $z$. Note that $j^k f$ determines $f$.
Now let $\mathfrak{A}$ be a closed semialgebraic subset [3] of the vector space $J^k(\mathbb{R}^m,\mathbb{R}^n)$ where $m = \dim(M)$. Suppose that $\mathfrak{A}$ is invariant under the right action of the group of diffeomorphisms $\mathbb{R}^m \to \mathbb{R}^m$, and of codimension $\geq m+2$ in $J^k(\mathbb{R}^m,\mathbb{R}^n)$. Let $\mathfrak{A}(M) \subset J^k(M,\mathbb{R}^n)$ consist of the jets which, in local coordinates about their source, belong to $\mathfrak{A}$. Let $\Gamma_{\neg\mathfrak{A}}(J^k(M,\mathbb{R}^n))$ be the space of smooth sections of the

vector bundle $J^k(M,\mathbb{R}^n) \to M$ which avoid $\mathfrak{A}(M)$. Let $\mathrm{map}_{\neg\mathfrak{A}}(M,\mathbb{R}^n)$ be the space of smooth maps $f\colon M \to \mathbb{R}^n$ whose jet prolongations avoid $\mathfrak{A}(M)$. Both are to be equipped with the Whitney $C^\infty$ topology.

**Theorem 5.4.** [36], [37]. *Suppose that $M$ is closed. Then with the above hypotheses on $\mathfrak{A}$, the jet prolongation map*

$$\mathrm{map}_{\neg\mathfrak{A}}(M,\mathbb{R}^n) \longrightarrow \Gamma_{\neg\mathfrak{A}}(J^k(M,\mathbb{R}^n))$$

*induces an isomorphism in cohomology with coefficients $\mathbb{Z}$. A corresponding statement holds for compact $M$ with boundary, with the convention that all smooth maps $M \to \mathbb{R}^n$ and all sections of $J^k(M,\mathbb{R}^n)$ in sight must agree near $\partial M$ with a prescribed $\varphi\colon M \to \mathbb{R}^n$ which has no $\mathfrak{A}$–singularities near $\partial M$.*

For an idea of how theorem 5.3 can be deduced from theorem 5.4, take $m = 3$, $n = 1$ and $k = 2$. Let $\mathfrak{A} \subset J^2(\mathbb{R}^3,\mathbb{R})$ be the set of 2–jets represented by germs

$$f\colon (\mathbb{R}^3, z) \to \mathbb{R}$$

which either have a nonzero value $f(z)$, or a nonzero first derivative at $z$, or a nondegenerate critical point at $z$. The codimension of $\mathfrak{A}$ is exactly $3 + 2$, the minimum of what is allowed in Vassiliev's theorem.
Change the definition of the "generalized surfaces" of type $\mathcal{W}$ given earlier by asking only that critical points of $f_x$ with critical value 0 be nondegenerate. In other words, require only that $f_x\colon E_x \to \mathbb{R}$ be Morse on a neighborhood of the compact set $f_x^{-1}(0)$. Change the definition of type $h\mathcal{W}$ generalized surfaces accordingly. These changes do not affect the homotopy types of $|\mathcal{W}|$ and $|h\mathcal{W}|$, by a shrinking argument similar to that given at the end of chapter 5. Note also that $\delta f_x$ in the definition of type $h\mathcal{W}$ ought to have been more correctly described as a section of the jet bundle $J^2(E_x,\mathbb{R}) \to E_x$. (After a choice of a Riemannian metric on $E_x$, an element of $J^k(E_x,\mathbb{R})$ with source $z \in E_x$ can be viewed as a polynomial function of degree $\le k$ on the tangent space of $E_x$ at $z$.) With these specifications and changes, theorem 5.3 begins to look like a special case of Vassiliev's theorem. It should however be seen as a generalization of a special case due to the fact that *families* of noncompact manifolds $E_x$ depending on a parameter $x \in X$ are involved. Vassiliev's theorem as stated above is about a "constant" compact manifold.

*Remarks concerning the proof of Vassiliev's theorem.* It is a complicated proof and the interested reader should, if possible, consult [36] as well as [37]. One of us (M.W.) has attempted to give an overview in [39], but this is already obsolete because of the following.
Vassiliev's proof uses a spectral sequence converging to the cohomology of the section space $\Gamma_{\neg\mathfrak{A}}(J^k(M,\mathbb{R}^n))$, and elaborate transversality and interpolation arguments to show that it converges to the cohomology of $\mathrm{map}_{\neg\mathfrak{A}}(M,\mathbb{R}^n)$, too. The spectral sequence is well hidden in the final paragraphs of the proof and looks as if it might depend on a number of obscure choices. But Elmer Rees informed us recently, naming Vassiliev as the source of this information, that the spectral sequence, from the second page onwards, does not depend on obscure choices and agrees with a spectral sequence of "generalized Eilenberg–Moore type", discovered already in 1972 by D.Anderson [2]. Anderson intended it as a spectral sequence converging to the (co)homology of a space of maps $X \to Y$. Here $X$ is a finite dimensional $CW$–space and $Y$ is a $\dim(X)$–connected space. (There is a version

for based spaces, too; the case where $X = \mathbb{S}^1$ and all maps are based is the standard Eilenberg–Moore spectral sequence [12], [30].) Vassiliev needs a variation where the space of maps is replaced by a space of sections of a certain bundle on $M$ whose fibers are $\dim(M)$–connected. The bundle is, of course, $\Gamma_{\neg\mathfrak{A}}(J^k(M,\mathbb{R}^n)) \to M$.
In conclusion, anybody wanting to understand Vassiliev's proof really well should try to understand the Anderson–Eilenberg–Moore spectral sequence for mapping spaces first. Anderson's article [2] is an announcement, but detailed proofs can be found in [4].

## 6. Stratifications and homotopy colimit decompositions

The developments in the previous section essentially reduce the proof of theorem 3.4 to the assertion that the homotopy fiber of the inclusion map $|\mathcal{W}| \to |\mathcal{W}_{\mathrm{loc}}|$ in diagram $(*)$ is homotopy equivalent to $\mathbb{Z} \times B\Gamma^+_{\infty,1+1}$. The proof of that assertion in [20] takes up many pages and relies mainly on compatible decompositions of $|\mathcal{W}|$ and $|\mathcal{W}_{\mathrm{loc}}|$ into manageable pieces. There is no point in repeating the details here. But there is a point in providing some motivation for the decompositions. The motivation which we propose here (very much "a posteriori") is almost perpendicular to the hard work involved in establishing the decompositions, and so does not overlap very much with anything in [20]. As a motivation for the motivation, we shall begin by describing the decompositions (without, of course, constructing them).

**Definition 6.1.** [31]. Let $\mathcal{F}$ be a covariant functor from a small category $\mathcal{C}$ to spaces. The *transport category* $\mathcal{C}\int\mathcal{F}$ is the topological category where the objects are the pairs $(c,x)$ with $c \in \mathrm{ob}(\mathcal{C})$ and $x \in \mathcal{F}(c)$, and where a morphism from $(c,x)$ to $(d,y)$ is a morphism $g\colon c \to d$ in $\mathcal{C}$ such that $\mathcal{F}(g)\colon \mathcal{F}(c) \to \mathcal{F}(d)$ takes $x$ to $y$. Thus $\mathrm{ob}(\mathcal{C}\int\mathcal{F})$ is the space $\coprod_c \mathcal{F}(c)$ and the morphism space $\mathrm{mor}(\mathcal{C}\int\mathcal{F})$ is the pullback of

$$\mathrm{ob}(\mathcal{C}\textstyle\int\mathcal{F}) \longrightarrow \mathrm{ob}(\mathcal{C}) \xleftarrow{\text{source}} \mathrm{mor}(\mathcal{C})\,.$$

The *homotopy colimit* of $\mathcal{F}$ is the classifying space of the topological category $\mathcal{C}\int\mathcal{F}$. Notation: $\operatorname{hocolim}\mathcal{F}$ , $\operatorname*{hocolim}_{\mathcal{C}} \mathcal{F}$ , $\operatorname*{hocolim}_{c \text{ in } \mathcal{C}} \mathcal{F}(c)$.

**Remarks 6.2.** If $\mathcal{C}$ has only one object, then $\mathcal{C}$ is a monoid, $\mathcal{F}$ amounts to a space with an action of the monoid, and $\operatorname{hocolim}\mathcal{F}$ is the Borel construction.
The variance of $\mathcal{F}$ is not important; if $\mathcal{F}$ is a contravariant functor from $\mathcal{C}$ to spaces, replace $\mathcal{C}$ by $\mathcal{C}^{\mathrm{op}}$ in the above definition. In that situation it is still customary to write $\operatorname{hocolim}_{\mathcal{C}}\mathcal{F}$ for the homotopy colimit.

**Definition 6.3.** Let $\mathcal{K}$ be the discrete category defined as follows. An object of $\mathcal{K}$ is a finite set $S$ with a map to $\{0,1,2,3\}$. A morphism from $S$ to $T$ in $\mathcal{K}$ consists of an injection $f\colon S \to T$ over $\{0,1,2,3\}$, and a map $\varepsilon$ from $T \smallsetminus f(S)$ to $\{-1,+1\}$. The composition of $(f_1,\varepsilon_1)\colon S \to T$ with $(f_2,\varepsilon_2)\colon R \to S$ is $(f_1f_2,\varepsilon_3)\colon R \to T$ where $\varepsilon_3(t) = \varepsilon_1(t)$ if $t \notin f_1(S)$ and $\varepsilon_3 f_1(s) = \varepsilon_2(s)$ if $s \notin f_2(R)$.

The category $\mathcal{K}$ arises very naturally in the taxonomy of generalized surfaces of type $\mathcal{W}$ and $\mathcal{W}_{\mathrm{loc}}$. Let $(E_x, f_x)$ be a generalized surface of type $\mathcal{W}$ or $\mathcal{W}_{\mathrm{loc}}$. Then the set $\mathrm{crit}_0(f_x) = \mathrm{crit}(f_x) \cap f_x^{-1}(0)$ is a finite set with a map to $\{0,1,2,3\}$ given by the Morse index. In other words it is an object of $\mathcal{K}$. In view of this, we expand our earlier list of generalized surface types by adding the following sub–types $\mathcal{W}_S$ and $\mathcal{W}_{\mathrm{loc},S}$ of types $\mathcal{W}$ and $\mathcal{W}_{\mathrm{loc}}$, respectively, for a fixed object $S$ of $\mathcal{K}$.

| | |
|---|---|
| $\mathcal{W}_S$ | oriented smooth 3–manifold $E_x$ with proper smooth Morse function $f_x\colon E_x \to \mathbb{R}$ and an isomorphism $S \to \mathrm{crit}_0(f_x)$ in $\mathcal{K}$ |
| $\mathcal{W}_{\mathrm{loc},S}$ | oriented smooth 3–manifold $E_x$ with smooth Morse function $f_x\colon E_x \to \mathbb{R}$ such that $f_x|\mathrm{crit}(f_x)$ is proper, and an isomorphism $S \to \mathrm{crit}_0(f_x)$ in $\mathcal{K}$ |

The classifying spaces for the corresponding families (which are, as usual, submersions) are denoted $|\mathcal{W}_S|$ and $|\mathcal{W}_{\mathrm{loc},S}|$, respectively. The promised decompositions of $|\mathcal{W}|$ and $|\mathcal{W}_{\mathrm{loc}}|$ can now be described loosely as follows.

**Theorem 6.4.** $|\mathcal{W}| \simeq \underset{S \text{ in } \mathcal{K}}{\mathrm{hocolim}}\ |\mathcal{W}_S|$ *and* $|\mathcal{W}_{\mathrm{loc}}| \simeq \underset{S \text{ in } \mathcal{K}}{\mathrm{hocolim}}\ |\mathcal{W}_{\mathrm{loc},S}|$.

Implicit in these formulae is the claim that $|\mathcal{W}_S|$ and $|\mathcal{W}_{\mathrm{loc},S}|$ are contravariant functors of the variable $S$ in $\mathcal{K}$. A rigorous verification would take up much space, and does take up much space in [20], but the true reasons for this functoriality are not hard to understand.

Fix a morphism $(g,\varepsilon)\colon S \to T$ in $\mathcal{K}$. Let $(E_x, f_x)$ be a generalized surface of type $\mathcal{W}_T$ or $\mathcal{W}_{\mathrm{loc},T}$, so that $\mathrm{crit}_0(f_x)$ is identified with $T$. Choose a smooth function $\psi\colon E_x \to \mathbb{R}$ with support in a small neighborhood of $\mathrm{crit}_0(f_x)$ such that $\psi$ equals $\varepsilon$ near points of $\mathrm{crit}_0(f_x) \cong T$ not in $g(S)$, and equals 0 near the remaining points of $\mathrm{crit}_0(f_x)$. Then for all sufficiently small $c > 0$, the function $f_x + c\psi$ is Morse and has exactly the same critical points as $f_x$. But the values of $f_x + c\psi$ on the critical points differ from those of $f_x$, with the result that $(E_x, f_x + c\psi)$ is a generalized surface of type $\mathcal{W}_S$ or $\mathcal{W}_{\mathrm{loc},S}$ as appropriate. The procedure generalizes to families and so induces maps

$$|\mathcal{W}_T| \longrightarrow |\mathcal{W}_S|\ , \qquad |\mathcal{W}_{\mathrm{loc},T}| \longrightarrow |\mathcal{W}_{\mathrm{loc},S}|\,.$$

Theorem 6.4 in its present raw state can be deduced from a *recognition principle* for homotopy colimits over certain categories. In the special case when the category is a group $G$, the recognition principle is well known and states the following.

*Suppose that $Y$ is the total space of a fibration $p\colon Y \to BG$.*
*Then $Y \simeq X_{hG}$ for some $G$–space $X$ such that $X \simeq p^{-1}(\star)$.*

(See remark 6.2, and for the proof let $X$ be the pullback along $p$ of the universal cover of $BG$.) In the general setting, the indexing category is an $EI$–category, that is, a category in which every Endomorphism is an Isomorphism. The category $\mathcal{K}$ is an example of an $EI$–category. Groupoids and posets are also extreme examples of $EI$–categories. The opposite category of any $EI$–category is an $EI$–category. $EI$–categories have something to do with stratified spaces, which justifies the following excursion.

**Definition 6.5.** A *stratification* of a space $Z$ is a locally finite partition of $Z$ into locally closed subsets, the *strata*, such that the closure of each stratum in $Z$ is a union of strata.

**Example 6.6.** Let $\mathcal{C}$ be a small EI–category. For each isomorphism class $[C]$ of objects in $\mathcal{C}$, we define a locally closed subset $B\mathcal{C}_{[C]}$ of the classifying space $B\mathcal{C}$, as follows. A point $x \in B\mathcal{C}$ is in $B\mathcal{C}_{[C]}$ if the unique cell of $B\mathcal{C}$ containing $x$ corresponds to a diagram

$$C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_k$$

without identity arrows, where $C_0$ is isomorphic to $C$. (Remember that $B\mathcal{C}$ is a CW–space, with one cell for each diagram $C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_k$ as above.) Then $B\mathcal{C}$ is stratified, with one stratum $B\mathcal{C}_{[C]}$ for each isomorphism class $[C]$. The closure of the stratum $B\mathcal{C}_{[C]}$ is the union of all strata $B\mathcal{C}_{[D]}$ for objects $D$ which admit a morphism $D \to C$.
To be even more specific, we can take $\mathcal{C} = \mathcal{K}^{\mathrm{op}}$. The isomorphism types of objects in $\mathcal{K}^{\mathrm{op}}$ correspond to quadruples $(n_0, n_1, n_2, n_3)$ of non–negative integers. The stratum of $B\mathcal{K}^{\mathrm{op}}$ corresponding to such a quadruple turns out to have a normal vector bundle in $B\mathcal{K}^{\mathrm{op}}$, of fiber dimension $n_0+n_1+n_2+n_3$; hence the stratum can be said to have codimension $n_0+n_1+n_2+n_3$. Its closure is the union of all strata corresponding to quadruples $(m_0, m_1, m_2, m_3)$ where $m_i \geq n_i$. There is a unique open stratum, corresponding to the object $\emptyset$ of $\mathcal{K}^{\mathrm{op}}$ or the quadruple $(0, 0, 0, 0)$.

*Digression.* The stratification of $B\mathcal{K}^{\mathrm{op}}$ just described can be used to determine the homotopy type of $B\mathcal{K}^{\mathrm{op}}$, roughly as follows. Let $f\colon X \to B\mathcal{K}^{\mathrm{op}}$ be a map, where $X$ is a smooth manifold. Up to a homotopy, $f$ is "transverse" to the strata of codimension $> 0$. Then the union of the inverse images of these codimension $> 0$ strata is the image of a proper smooth codimension 1 immersion $M \to X$ with trivialized normal line bundle, with transverse self–intersections, and with a map $M \to \{0, 1, 2, 3\}$. The construction can be reversed, i.e., such an immersion determines a homotopy class of maps $X \to B\mathcal{K}^{\mathrm{op}}$. In this sense, $B\mathcal{K}^{\mathrm{op}}$ classifies (up to concordance) proper smooth codimension one immersions with trivialized normal bundle and with a map from the source $M$ to $\{0, 1, 2, 3\}$. It follows that

$$B\mathcal{K}^{\mathrm{op}} \simeq Q\mathbb{S}^1 \times Q\mathbb{S}^1 \times Q\mathbb{S}^1 \times Q\mathbb{S}^1$$

because $Q\mathbb{S}^1 = \Omega^\infty\Sigma^\infty\mathbb{S}^1$ is known to classify proper smooth codimension 1 immersions with trivialized normal bundle [38].

**Definition 6.7.** Let $Z$ be a stratified space. A path $\gamma\colon [0, 1] \to Z$ is *nonincreasing* if, for each $t \in [0, 1]$, the set $\gamma[0, t]$ is contained in the closure of the stratum which contains $\gamma(t)$. A homotopy of maps $(h_t\colon X \to Z)_{t\in[0,1]}$, where $X$ is some space, is *nonincreasing* if, for each $x \in X$, the path $t \mapsto h_t(x)$ is nonincreasing.

*Remark.* For a nonincreasing path $\gamma$, the depth, complexity, etc. of the stratum containing $\gamma(t)$ is a nonincreasing function of $t$.

**Definition 6.8.** Let $p\colon Y \to Z$ be a map, where $Z$ is stratified. Say that $p$ is a *downward fibration* if it has the homotopy lifting property for nonincreasing homotopies. That is, given a nonincreasing homotopy $(h_t\colon X \to Z)_{t\in[0,1]}$ and a map $g_0\colon X \to Y$ such that $pg_0 = h_0$, there exists a homotopy $(g_t\colon X \to Y)_{t\in[0,1]}$ such that $pg_t = h_t$ for all $t \in [0, 1]$.

**Pre-theorem 6.9.** *Let $\mathcal{C}$ be an EI–category. Stratify $B\mathcal{C}$ as in example 6.6. Let $Y$ be a space and let $p\colon Y \to B\mathcal{C}$ be a downward fibration. Then*

$$Y \simeq \operatorname*{hocolim}_{c \text{ in } \mathcal{C}} \mathcal{F}(c)$$

*where $\mathcal{F}$ is a covariant functor from $\mathcal{C}$ to spaces such that $\mathcal{F}(c) \simeq p^{-1}(c)$ for all objects $c$ of $\mathcal{C}$, alias vertices of $B\mathcal{C}$.*

This is the recognition principle (no proof offered for lack of time and space). It has an obvious weakness: the functor $\mathcal{F}$ is not sufficiently determined by the conditions $\mathcal{F}(c) \simeq p^{-1}(c)$. But then it is meant as a principle, a rule of thumb.

In any case we should apply it with $Y = |\mathcal{W}|$ or $Y = |\mathcal{W}_{\mathrm{loc}}|$ and $\mathcal{C} = \mathcal{K}^{\mathrm{op}}$, the opposite of $\mathcal{K}$. There is a problem with that plan. Explicit descriptions of $|\mathcal{W}|$ and $|\mathcal{W}_{\mathrm{loc}}|$ have not yet been given (in this paper). Instead, we have highfalutin characterizations of $|\mathcal{W}|$ and $|\mathcal{W}_{\mathrm{loc}}|$ as classifying spaces for certain families. The modified plan is, therefore, to move $B\mathcal{C} = B\mathcal{K}^{\mathrm{op}}$ to the same highfalutin level, and to verify the hypothesis of pretheorem 6.9 at that level.
This leads us to the interesting question: *What does the classifying space of a category $\mathcal{C}$ classify ?* There is no doubt that the question has many correct answers. One such answer is given in [20, 4.1.2]. This is essentially identical with an answer known to tom Dieck (but possibly attributed to G. Segal) in the early 70's, according to unpublished lecture notes for which we are indebted to R. Vogt. Moerdijk [23] has a more streamlined answer, and many generalizations of the question, too. The following proposal is inspired by a passage in Moerdijk's book, but is apparently not identical with (a special case of) his answer and if it should fail badly the responsibility is ours.
*Terminology.* A $\mathcal{C}$–set is a functor from $\mathcal{C}^{\mathrm{op}}$ to sets. The category of $\mathcal{C}$–sets shares many good properties with the category of sets. (It is a topos.) In particular, we can talk about sheaves of $\mathcal{C}$–sets on a space. A $\mathcal{C}$–set is representable if it is isomorphic to one of the form $c \mapsto \mathrm{mor}_{\mathcal{C}}(c, c_0)$ for a fixed object $c_0$ in $\mathcal{C}$.

**Pre-theorem 6.10.** *The classifying space $B\mathcal{C}$ classifies sheaves of $\mathcal{C}$–sets whose stalks are representable.*

*Remark.* Traditionally there are two equivalent definitions of the notion "sheaf" on a space $X$. According to one of them, a sheaf is a contravariant functor from the poset of open sets of $X$ to sets, subject to a gluing condition. According to the other, a sheaf on $X$ is an étale map to $X$. While the first point of view is better for processing most of the interesting examples, the second one is better for showing that sheaves behave contravariantly (can be "pulled back"). This carries over to sheaves of $\mathcal{C}$–sets.
The classification of the sheaves in pretheorem 6.10 is up to concordance. Two sheaves $\mathcal{G}_0$, $\mathcal{G}_1$ on $X$ as in the pretheorem are *concordant* if there exists a sheaf on $X \times [0,1]$, as in the pretheorem, whose restrictions to $X \times \{0\} \cong X$ and $X \times \{1\} \cong X$ are isomorphic to $\mathcal{G}_0$ and $\mathcal{G}_1$, respectively. The claim is that, for "most" spaces $X$, there is a natural bijection from the set of homotopy classes $[X, B\mathcal{C}]$ to the set of concordance classes of sheaves of $\mathcal{C}$–sets with representable stalks on $X$.

**Example 6.11.** Let $(\pi, f)$ be a family of generalized surfaces of type $\mathcal{W}$ on a smooth $X$. That is, $\pi\colon E \to X$ is a smooth submersion with oriented 3–dimensional fibers, $f\colon E \to \mathbb{R}$ is a smooth map such that $(\pi, f)\colon E \to X \times \mathbb{R}$ is proper, and the restrictions $f_x = f|E_x$ of $f$ to the fibers of $\pi$ are Morse functions. With these data, we can associate a sheaf $\mathcal{I}_{(\pi,f)}$ of $\mathcal{K}^{\mathrm{op}}$–sets on $X$. Namely, for an open subset $U$ of $X$ and an object $S$ of $\mathcal{K}$, let $\mathcal{I}_{(\pi,f)}(U)(S)$ be the subset of

$$\prod_{x \in U} \mathrm{mor}_{\mathcal{K}}(\mathrm{crit}_0(f_x), S)$$

consisting of the elements for which the adjoint map from $\bigcup_{x \in U} \mathrm{crit}_0(f_x) \subset E$ to $S$ is continuous, and the resulting sign function from a subset of $U \times S$ to $\{\pm 1\}$ is also continuous. Then $\mathcal{I}_{(\pi,f)}(U)(S)$ is a covariant functor of $S$ in $\mathcal{K} = (\mathcal{K}^{\mathrm{op}})^{\mathrm{op}}$ and a contravariant functor of the variable $U$, as it should be. The stalk at $x \in X$ is easily identified with the functor $S \mapsto \mathrm{mor}_{\mathcal{K}}(\mathrm{crit}_0(f_x), S)$. It is obviously representable as a functor on $\mathcal{K}^{\mathrm{op}}$.

The construction of $\mathcal{I}_{(\pi,f)}$ in example 6.11 works equally well for a family of generalized surfaces of type $\mathcal{W}_{\mathrm{loc}}$.

Now theorem 6.4 can be understood as a special case of (something analogous to) the recognition principle, pretheorem 6.9. Take $\mathcal{C} = \mathcal{K}^{\mathrm{op}}$ and $Y = |\mathcal{W}|$ or $Y = |\mathcal{W}_{\mathrm{loc}}|$ in pretheorem 6.9. There are no explicit maps $|\mathcal{W}| \to B\mathcal{K}^{\mathrm{op}}$ or $|\mathcal{W}_{\mathrm{loc}}| \to B\mathcal{K}^{\mathrm{op}}$ to work with. But there is instead the procedure of example 6.11 which from every family $(\pi, f)$ of the sort classified by $|\mathcal{W}|$ or $|\mathcal{W}_{\mathrm{loc}}|$ constructs a sheaf $\mathcal{I}_{(\pi,f)}$ of the sort classified by $B\mathcal{K}^{\mathrm{op}}$. The "downward fibration" condition in pretheorem 6.9 can be stated and proved in this setting.
In more detail, for the case of $|\mathcal{W}|$, fix a smooth manifold $X$ and a sheaf $\mathcal{H}$ of $\mathcal{K}^{\mathrm{op}}$–sets on $X \times [0,1]$ with representable stalks. Assume that $\mathcal{H}$ is *nonincreasing*. This means simply that, for every $x \in X$, the function which to $t \in [0,1]$ assigns the cardinality of the representing object for the stalk at $(x,t)$ is nonincreasing. Assume further that the restriction of $\mathcal{H}$ to $X \times \{0\}$ is identified with $\mathcal{I}_{(\pi,f)}$ for a family $(\pi, f)$ on $X \times \{0\}$, as in example 6.11. Then we can extend that family to a family $(\psi, g)$ on $X \times [0,1]$, and the isomorphism of sheaves to an isomorphism of $\mathcal{I}_{(\psi,g)}$ with $\mathcal{H}$. The verification is left to the reader.

## 7. Final touches

The guiding idea for this chapter is that, because of theorem 6.4, we should be able to understand the homotopy fiber(s) of $|\mathcal{W}| \to |\mathcal{W}_{\mathrm{loc}}|$ by understanding the homotopy fibers of $|\mathcal{W}_S| \to |\mathcal{W}_{\mathrm{loc},S}|$ for each object $S$ in $\mathcal{K}$. The underpinning for this strategy is the following general fact. (Notation: "$\mathrm{hofiber}_z(f)$" is short for the homotopy fiber over a point $z$ in the target of a map $f$.)

**Proposition 7.1.** *Let $\mathcal{C}$ be a small category and let $\mathcal{F}_1$, $\mathcal{F}_2$ be functors from $\mathcal{C}$ to the category of spaces. Let $u\colon \mathcal{F}_1 \to \mathcal{F}_2$ be a natural transformation. Suppose that, for every object morphism $g\colon c \to d$ and every $z \in \mathcal{F}_2(c)$, the map*

$$\mathrm{hofiber}_z\,(\mathcal{F}_1(c) \to \mathcal{F}_2(c)) \xrightarrow{\;g_*\;} \mathrm{hofiber}_{u_*(z)}\,(\mathcal{F}_1(d) \to \mathcal{F}_2(d))$$

*induced is a homotopy equivalence (resp., induces an isomorphism in integral homology). Then, for any object $c$ in $\mathcal{C}$ and $z \in \mathcal{F}_2(c)$, the inclusion*

$$\mathrm{hofiber}_z\,(\mathcal{F}_1(c) \to \mathcal{F}_2(c)) \longrightarrow \mathrm{hofiber}_z\,(\mathrm{hocolim}\,\mathcal{F}_1 \to \mathrm{hocolim}\,\mathcal{F}_2)$$

*is a homotopy equivalence (resp., induces an isomorphism in integral homology).*

We now have to ask whether the hypotheses of this proposition are satisfied or "nearly satisfied" in the case where $\mathcal{C}$ is (equivalent to) $\mathcal{K}^{\mathrm{op}}$ and $u\colon \mathcal{F}_1 \to \mathcal{F}_2$ is given by the inclusions $|\mathcal{W}_S| \to |\mathcal{W}_{\mathrm{loc},S}|$ for $S$ in $\mathcal{K}^{\mathrm{op}}$.

**Lemma 7.2.** *The space $|\mathcal{W}_{\mathrm{loc},S}|$ is a classifying space for oriented 3–dimensional Riemannian vector bundles $V$ on $S$ equipped with the following extra structure: an orthogonal splitting $V \cong V(+) \oplus V(-)$, where the fiber dimension function of $V(-)$ agrees with the structure map $S \to \{0,1,2,3\}$.*

*Idea of proof.* A generalized surface $(E_x, f_x)$ of type $\mathcal{W}_{\mathrm{loc},S}$ is canonically concordant to $(V, f_x|V)$ for any open neighborhood $V$ of $\mathrm{crit}_0(f_x) \cong S$. Take $V$ to be a standard tubular neighborhood of $S$, so that the retraction $V \to S$ comes with a vector bundle structure. By Morse theory, there is no substantial loss of information in replacing $f_x|V$ by the "total Hessian" of $f_x$ , which is a nondegenerate symmetric form on $V$. A choice of an orthogonal splitting of $V$ into a positive definite part and a negative definite part for the Hessian can be added, because that is a contractible choice. By changing the sign of the Hessian on the negative definite summand, we obtain a Riemannian structure on $V$. □

**Lemma 7.3.** *The space $|\mathcal{W}_S|$ is a classifying space for bundles of smooth closed oriented surfaces, where each fiber $F$ is equipped with "surgery data" as follows:*

- *a 3–dimensional vector bundle $V$ on $S$, etc., as in lemma 7.2;*
- *a smooth orientation preserving embedding $e$ of $\mathbb{D}(V(+)) \times_S \mathbb{S}(V(-))$ in $F$, where $\mathbb{D}(\dots)$ and $\mathbb{S}(\dots)$ denotes unit disk and unit sphere bundles.*

*Idea of proof.* In the definition of a generalized surface $(E_x, f_x)$ of type $\mathcal{W}_S$, add the condition $\mathrm{crit}_0(f_x) = \mathrm{crit}(f_x)$, so that critical values other than 0 are forbidden. A shrinking argument similar to all the previous shrinking arguments in this paper shows that this change does not affect the homotopy type of the classifying space $|\mathcal{W}_S|$. With the new condition $\mathrm{crit}_0(f_x) = \mathrm{crit}(f_x)$, however, a generalized surface $(E_x, f_x)$ of type $\mathcal{W}_S$ can be described as the (long) trace of $|S|$ simultaneous surgeries on the genuine smooth oriented surface $f_x^{-1}(c)$ for fixed $c < 0$. The simultaneous surgeries are in the usual way determined by disjoint embeddings of certain thickened spheres (labelled by the elements of $S$) in the surface $f_x^{-1}(c)$. □

**Corollary 7.4.** *The homotopy fiber of $|\mathcal{W}_S| \to |\mathcal{W}_{\mathrm{loc},S}|$ over any point $z$ in $|\mathcal{W}_{\mathrm{loc},S}|$ is a classifying space for bundles of compact smooth oriented surfaces with a prescribed (oriented) boundary depending on $S$ and $z$.*

*Outline of proof.* The choice of $z$ amounts to a choice of a Riemannian vector bundle $V$ on $S$ with splitting etc., as in lemma 7.2. To obtain a correct description of the homotopy fiber, simply fix $V$ etc. in the re–definition of $|\mathcal{W}_S|$ given in lemma 7.3. This fixes the source of the codimension zero embedding $e$. Hence the information carried by the surface $F$ and the embedding $e$ is carried by the closure of $F \smallsetminus \mathrm{im}(e)$, and the identification of its boundary with $\mathbb{S}(V(+)) \times_S \mathbb{S}(V(-))$. □

It is obvious how the homotopy fibers in corollary 7.4 depend on $z$ alias $V$. The dependence on $S$ is more interesting because we can vary by morphisms in $\mathcal{K}$ which are not isomorphisms. It suffices to describe the dependence in the case of a morphism $(g, \varepsilon)\colon R \to S$ in $\mathcal{K}$ where $g$ is an inclusion and $S \smallsetminus R$ has a single element $s$. Let $z \in |\mathcal{W}_{\mathrm{loc},S}|$ correspond to a vector bundle $V$ on $S$, etc., as in 7.2. Then the image $y \in |\mathcal{W}_{\mathrm{loc},R}|$ of $z$ under $(g, \varepsilon)^*$ corresponds to $V|R = V \smallsetminus V_s$.

**Lemma 7.5.** *The map induced by $(g, \varepsilon)$ from the homotopy fiber of $|\mathcal{W}_S| \to |\mathcal{W}_{\mathrm{loc},S}|$ over $z$ to the homotopy fiber of $|\mathcal{W}_R| \to |\mathcal{W}_{\mathrm{loc},R}|$ over $y$ is given by a gluing construction $\sqcup_{\partial L} L$, applied to the surfaces featuring in corollary 7.4, where*

$$L \quad = \quad \begin{cases} \mathbb{D}(V_s(+)) \times \mathbb{S}(V_s(-)) & \text{if } \varepsilon(s) = +1 \\ \mathbb{S}(V_s(+)) \times \mathbb{D}(V_s(-)) & \text{if } \varepsilon(s) = -1. \end{cases}$$

The transition maps described in this lemma do not induce homology isomorphisms in general (i.e., do not satisfy the conditions of proposition 7.1), but in a sense they come close to that. Indeed they are maps of the type considered in the Harer–Ivanov stability theorem 1.5. The remaining difficulty, from this point of view, is therefore that the surfaces featuring in corollary 7.4 need not be connected and of large genus. Fortunately it is possible to make some changes in the decomposition $|\mathcal{W}| \simeq \text{hocolim}_S |\mathcal{W}_S|$ so that corollary 7.4 comes out "right", i.e., with something resembling the phrase *connected and of large genus* in it. (A welcome side–effect of these changes is that $|\mathcal{W}_\emptyset|$ metamorphoses into $\mathbb{Z} \times B\Gamma_{\infty,1+1}$.) These adjustments occupy the final chapters of [20].

## References


1. J. F. Adams, *Infinite loop spaces*, Annals of Math. Studies, vol. 90, Princeton Univ. Press, 1978.
2. D. W. Anderson, *A generalization of the Eilenberg–Moore spectral sequence*, Bull. Amer. Math. Soc. **78** (1972), 784–786.
3. R. Benedetti and J.-J. Risler, *Real algebraic and semi–algebraic sets*, Actualités mathématiques, Hermann, 1990.
4. A. K. Bousfield, *On the homology spectral sequence of a cosimplicial space*, Amer. J. Math. **109** (1987), 361–394.
5. T. Bröcker and K. Jänich, *Introduction to differential topology*, Camb. Univ. Press, 1982, German edition Springer-Verlag (1973).
6. E. Brown, *Cohomology theories*, Ann. of Math. **75** (1962), 467–484, correction in Ann. of Math. 78 (1963).
7. S. Buoncristiano, C. Rourke, and B. Sanderson, *A geometric approach to homology theory*, Lond. Math. Soc. Lecture Note ser., vol. 18, Camb. Univ. Press, 1976.
8. S. S. Chern, *An elementary proof of the existence of isothermal parameters on a surface*, Proc. Amer. Math. Soc. **6** (1955), 771–782.
9. D.Quillen, *Elementary proofs of some results of cobordism theory using Steenrod operations*, Advances in Math. **7** (1971), 29–56.
10. C. J. Earle and J. Eells, *A fibre bundle description of Teichmüller theory*, J. Differential Geom. **3** (1969), 19–43.
11. C. J. Earle and A. Schatz, *Teichmüller theory for surfaces with boundary*, J. Differential Geom. **4** (1970), 169–185.
12. S. Eilenberg and J. C. Moore, *Homology and fibrations. I. Coalgebras, cotensor product and its derived functors*, Comment. Math. Helv. **40** (1966), 199–236.
13. S. Galatius, *Mod p homology of the stable mapping class group*, Topology **43** (2004), 1105–1132.
14. A. Haefliger, *Lectures on the theorem of Gromov*, Proc. of 1969/70 Liverpool Singularities Symp., Lecture Notes in Math., vol. 209, Springer, 1971, pp. 128–141.
15. J. L. Harer, *Stability of the homology of the mapping class groups of oriented surfaces*, Ann. of Math. **121** (1985), 215–249.
16. A. Hatcher and W. Thurston, *A presentation for the mapping class group of a closed orientable surface*, Topology **19** (1980), 221–237.
17. N. V. Ivanov, *Stabilization of the homology of the Teichmueller modular groups*, Algebra i Analiz **1** (1989), 110–126, translation in: Leningrad Math. J. 1 (1990) 675–691.
18. ______, *On the homology stability for Teichmüller modular groups: closed surfaces and twisted coefficients*, Mapping class groups and moduli spaces of Riemann surfaces (Göttingen/Seattle 1991), Contemporary Mathematics, vol. 150, Amer. Math. Soc., 1993, pp. 149–194.
19. I. Madsen and U. Tillmann, *The stable mapping class group and* $Q(\mathbb{C}P^\infty)$, Invent. Math. **145** (2001), 509–544.
20. I. Madsen and M. Weiss, *The stable moduli space of Riemann surfaces: Mumford's conjecture*, preprint, arXiv:math.AT/0212321, 2002.

21. D. McDuff and G. Segal, *Homology fibrations and the "group–completion" theorem*, Invent. Math. **31** (1976), 279–284.
22. E. Miller, *The homology of the mapping class group*, J. Diff. Geom. **24** (1986), 1–14.
23. I. Moerdijk, *Classifying spaces and classifying topoi*, Lecture Notes in Math., vol. 1616, Springer, 1995.
24. S. Morita, *Characteristic classes of surface bundles*, Bull. Amer. Math. Soc. **11** (1984), 386–388.
25. ______, *Characteristic classes of surface bundles*, Invent. Math. **90** (1987), 551–577.
26. D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, Aritmetic and Geometry, II, Progr. in Math., vol. 36, Birkhäuser, 1983, pp. 271–328.
27. A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. of Math. **65** (1957), 391–404.
28. A. Phillips, *Submersions of open manifolds*, Topology **6** (1967), 170–206.
29. J. Powell, *Two theorems on the mapping class group of a surface*, Proc. Amer. Math. Soc. **68** (1978), 347–359.
30. D. L. Rector, *Steenrod operations in the Eilenberg–Moore spectral sequence*, Comment. Math. Helv. **45** (1970), 540–552.
31. G. Segal, *Classifying spaces and spectral sequences*, Inst. Hautes Etudes Sci. Publ Math. **34** (1968), 105–112.
32. G. B. Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312.
33. E. H. Spanier, *Algebraic topology*, McGraw–Hill, New York, 1966.
34. U. Tillmann, *On the homotopy of the stable mapping class group*, Invent. Math. **130** (1997), 257–275.
35. ______, *A splitting for the stable mapping class group*, Math. Proc. Camb. Phil. Soc. **127** (1999), 55–56.
36. V. Vassiliev, *Topology of spaces of functions without complicated singularities*, Funktsional. Anal. i Prilozhen. **93** (1989), 24–36, Engl. translation in Funct. Analysis Appl. 23 (1989), 266–286.
37. ______, *Complements of Discriminants of Smooth Maps: Topology and Applications*, Transl. of Math. Monographs, vol. 98, Amer. Math. Soc., 1994 (1992), revised edition.
38. P. Vogel, *Cobordisme d'immersions*, Ann. Sci. Ecole Norm. Sup. **7** (1974), 317–357.
39. M. Weiss, *Cohomology of the stable mapping class group*, Topology, Geometry and Quantum Field Theory: Proc. of 2002 Oxf. Symp. in honour of G. Segal's 60th birthday, Cambridge Univ. Press, 2004, ed. by U. Tillmann.

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