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THE MUMFORD CONJECTURE
[after Madsen and Weiss]

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1. INTRODUCTION

The Mumford conjecture concerns the cohomology of the moduli space $\mathcal{M}_g$ of smooth projective curves of genus g: Mumford constructed tautological classes κ_i, for $i \geq 1$, in the Chow ring $CH^i(\mathcal{M}_g)$ with rational coefficients, which yield a natural morphism of algebras $\mathbb{Q}[\kappa_i] \to CH^*(\mathcal{M})$, in which $CH^*(\mathcal{M})$ denotes the Chow ring of the moduli spaces, stabilized with respect to the genus. The conjecture asserts that the above morphism is an isomorphism [17, 8].

The conjecture can be reformulated in terms of the stable cohomology of the mapping class groups (or Teichmüller modular groups) Γ_g [5, 16]. The mapping class group Γ_g is the discrete group of isotopy classes of orientation-preserving diffeomorphisms of a smooth, oriented surface of genus g. The group cohomology $H^*(B\Gamma_g)$ of the mapping class groups stabilizes in a given degree for sufficiently large genus. The stable value identifies with the cohomology of the space $B\Gamma_\infty$, which is the homotopy colimit of the system of classifying spaces $B\Gamma_{g,2}$ of the mapping class groups of curves with two marked points, stabilized with respect to maps induced by group morphisms $\Gamma_{g,2} \to \Gamma_{g+1,2}$.

The moduli space $\mathcal{M}_g$ can be constructed, as an analytic space, as the quotient of the action of the group Γ_g upon Teichmüller space, $\mathcal{T}_g$. Teichmüller space is contractible and the action has finite isotopy groups, hence the Mumford conjecture can be restated in terms of the Mumford-Morita-Miller characteristic classes [14, 15], $\kappa_i \in H^{2i}(B\Gamma_\infty; \mathbb{Q})$.

Conjecture 1.1. — *The classes $\kappa_i \in H^{2i}(B\Gamma_\infty; \mathbb{Q})$ induce an isomorphism of algebras $\tilde{\alpha} : \mathbb{Q}[\kappa_i] \to H^*(B\Gamma_\infty; \mathbb{Q})$.*

The algebra $H^*(B\Gamma_\infty; \mathbb{Q})$ has a Hopf algebra structure, induced by a multiplicative structure of geometric origin on the classifying space $B\Gamma_\infty$. The classes κ_i are primitive and non-trivial, thus the morphism $\tilde{\alpha}$ is a monomorphism of Hopf algebras.

The space $B\Gamma_\infty$ has a structure which enriches the multiplicative structure; namely, the space $B\Gamma_\infty$ has a perfect fundamental group, hence the Quillen plus construction applies to yield a morphism $B\Gamma_\infty \to B\Gamma_\infty^+$, which induces an isomorphism in homology and such that $B\Gamma_\infty^+$ has trivial fundamental group. Tillmann [22] showed that the space $\mathbb{Z} \times B\Gamma_\infty^+$ is an infinite loop space, hence it represents the degree zero part of a generalized cohomology theory; the identification of the associated cohomology theory is a problem of *stable* homotopy theory.

The construction of the Mumford-Morita-Miller characteristic classes uses integration along the fibre of powers of the first Chern class of the orientation bundle of the universal oriented surface bundle. This can be interpreted in terms of the Gysin morphism, which is of topological origin, via the Pontrjagin-Thom construction. Madsen and Tillmann [11] constructed a morphism of infinite loop spaces

$$\alpha_\infty : \mathbb{Z} \times B\Gamma_\infty^+ \to \Omega^\infty(\mathbb{C}\mathbf{P}^\infty_{-\mathbf{1}})$$

which lifts the construction of $\tilde{\alpha}$. The infinite loop space $\Omega^\infty(\mathbb{C}\mathbf{P}^\infty_{-\mathbf{1}})$ is constructed from the Thom spectrum which is associated to the complements of the canonical line bundles on complex projective space.

The rational cohomology of the space $\Omega^\infty(\mathbb{C}\mathbf{P}^\infty_{-\mathbf{1}})$ is isomorphic to the rational cohomology of the space $\mathbb{Z} \times BU$, where BU denotes the classifying space of the infinite unitary group. The cohomology algebra $H^*(BU;\mathbb{Q})$ is isomorphic to the polynomial algebra $\mathbb{Q}[\kappa_i]$, where the classes κ_i can be taken to be Chern classes, hence the Mumford conjecture is implied by the following result, which is referred to as the generalized Mumford conjecture.

Theorem 1.2. — [12] *The morphism* $\alpha_\infty : \mathbb{Z} \times B\Gamma_\infty^+ \to \Omega^\infty(\mathbb{C}\mathbf{P}^\infty_{-\mathbf{1}})$ *is a homotopy equivalence.*

The cohomology of the space $\Omega^\infty(\mathbb{C}\mathbf{P}^\infty_{-\mathbf{1}})$ with coefficients in a finite field $\mathbb{F}_p$ has been calculated [4], using techniques of algebraic topology. The above theorem therefore yields a calculation of the stable cohomology of the mapping class groups $H^*(B\Gamma_\infty;\mathbb{F}_p)$, for any prime p.

1.1. Methods of proof

Madsen and Weiss reformulate the generalized Mumford conjecture using certain generalized bundle theories; these are local in nature and their classifying spaces can be constructed from realization spaces associated to sheaves of sets. In particular, they give an interpretation of a modification of the morphism α_∞ introduced in [11] as the realization of a morphism of sheaves.

Let $\mathfrak{X}$ denote the category of smooth manifolds, without boundary and with a countable basis and consider sheaves of sets on $\mathfrak{X}$. There is a natural notion of homotopy on the sections of a sheaf, termed *concordance*; if $\mathcal{F}$ is a sheaf and X is a smooth manifold, then concordance is an equivalence relation on the sections $\mathcal{F}(X)$, which is induced by elements of $\mathcal{F}(X \times \mathbb{R})$, in the usual way. The set of equivalence classes for the concordance relation is written as $\mathcal{F}[X]$. The contravariant functor $X \mapsto \mathcal{F}[X]$ is represented on the homotopy category of topological spaces by a space $|\mathcal{F}|$; namely $\mathcal{F}[X] \cong [X, |\mathcal{F}|]$, where the right hand side denotes homotopy classes of morphisms of topological spaces.

A morphism $f : \mathcal{E} \to \mathcal{F}$ of sheaves is defined to be a weak equivalence if the induced morphism $|f| : |\mathcal{E}| \to |\mathcal{F}|$ is a weak equivalence(1) between the representing spaces.

There are two principal techniques which are used to show that a morphism between sheaves is a weak equivalence: to exhibit explicit concordances so as to obtain an isomorphism of concordance classes or to use the relative surjectivity criterion of Proposition A.7 to show that a morphism is a weak equivalence.

The classifying space $B\Gamma_g$ classifies bundles with fibres which are closed oriented surfaces, hence the source of the morphism α_∞ is related to bundles of closed oriented surfaces. This motivates consideration of the sheaf $\mathcal{V}$ with sections over X the set of pairs (π, f), where $\pi : E \to X$ is a smooth submersion with 3-dimensional oriented fibres and $f : E \to \mathbb{R}$ is a smooth morphism such that (π, f) is a proper submersion. Ehresmann's fibration lemma implies that this is a bundle of smooth surfaces on $X \times \mathbb{R}$.

The definition of $\mathcal{V}$ can be weakened: let $h\mathcal{V}$ denote the sheaf with sections over X the set of pairs $(\pi, \hat{f})$, where $\pi : E \to X$ is as before and $\hat{f}$ is a smooth section of the fibrewise 1-jet bundle $J^1_\pi(E, \mathbb{R}) \to E$, subject to the condition that the morphism $(\pi, f) : E \to X \times \mathbb{R}$ is a proper submersion, where f denotes the underlying smooth map, $f : E \to \mathbb{R}$, of $\hat{f}$. There is a morphism of sheaves $\alpha : \mathcal{V} \to h\mathcal{V}$, induced by jet prolongation, which induces a morphism of topological spaces $|\alpha| : |\mathcal{V}| \to |h\mathcal{V}|$, which is related to the morphism α_∞.

These definitions generalize; namely it is expedient to allow mild fibrewise singularities over $X \times \mathbb{R}$, by considering smooth sections of the fibrewise 2-jet bundle $J^2_\pi(E, \mathbb{R}) \to E$ and permitting fibrewise critical points which are of Morse type. This gives sheaves $\mathcal{W}, h\mathcal{W}$, where $\mathcal{W}$ corresponds to the integrable situation, as above. Similarly, there are sheaves $\mathcal{W}_{\mathrm{loc}}, h\mathcal{W}_{\mathrm{loc}}$ which correspond to the local situation around the singular sets and these sheaves form a commutative diagram

$$\begin{tikzcd} \mathcal{V} \arrow[r] \arrow[d, "j^2_\pi"'] & \mathcal{W} \arrow[r] \arrow[d, "j^2_\pi"'] & \mathcal{W}_{\mathrm{loc}} \arrow[d, "j^2_\pi"] \\ h\mathcal{V} \arrow[r] & h\mathcal{W} \arrow[r] & h\mathcal{W}_{\mathrm{loc}}. \end{tikzcd} \tag{1}$$

The first main theorem of Vassiliev on the space of functions with moderate singularities is used to show the following Theorem, which motivates the strategy of proof.

Theorem 1.3. — *The morphism $j^2_\pi : \mathcal{W} \to h\mathcal{W}$ is a weak equivalence.*

This result is used in conjunction with the following, which is proved using bordism theory.

Theorem 1.4. —

1. *The morphism $j^2_\pi : \mathcal{W}_{\mathrm{loc}} \to h\mathcal{W}_{\mathrm{loc}}$ is a weak equivalence.*

(1) (induces an isomorphism on homotopy groups)

2. *The sequence of representing spaces* $|h\mathcal{V}| \to |h\mathcal{W}| \to |h\mathcal{W}_{\mathrm{loc}}|$ *is a homotopy fibre sequence*[2] *of infinite loop spaces.*
3. *There is a homotopy equivalence* $|h\mathcal{V}| \simeq \Omega^\infty(\mathbb{C}\mathbf{P}^\infty_{-1})$.

Let F denote the homotopy fibre of $|\mathcal{W}| \to |\mathcal{W}_{\mathrm{loc}}|$, then it follows formally from the homotopy invariance of the homotopy fibre construction that there is a homotopy equivalence $F \xrightarrow{\simeq} |h\mathcal{V}|$. Standard methods of homotopy theory imply that the generalized Mumford conjecture follows from:

Theorem 1.5. — *There exists a morphism* $\mathbb{Z} \times B\Gamma_\infty \to F$ *which induces an isomorphism in homology with integral coefficients.*

The proof of this theorem involves replacing the singularities inherent in $\mathcal{W}$ by ones in standard form and then stratifying by critical sheets; after stratification, the concordance relation is imposed by a homotopical gluing construction, the homotopy colimit over a suitable category. The proof of the theorem relies on foundational results from homotopy theory together with the homological stability results of Harer; in particular, the proof uses closed surfaces with boundary.

1.2. Approximations

Much of the material of [12] is developed for bundles of manifolds of arbitrary dimension, d, and with a general notion of orientation, the Θ-orientation. For the presentation of this text, the general notion of orientation has been suppressed and the integer d is usually taken to be two.

To avoid set-theoretic difficulties, [12] uses the notion of graphic morphisms with respect to a fixed set in the definitions of the sheaves which are considered; moreover set-theoretic caveats are required in various proofs. All such details have been suppressed in this text.

2. MAPPING CLASS GROUPS

2.1. Orientation-preserving diffeomorphisms

Let F be a smooth, compact, oriented surface with boundary ∂F, then F is classified, up to diffeomorphism, by its genus g and the number b of boundary components; write $F_{g,b}$ for a representative of the diffeomorphism class.

The topological group of orientation-preserving diffeomorphisms of F which fix the boundary is written $\mathrm{Diff}^\circ(F;\partial F)$ and ${\mathrm{Diff}^\circ}_e(F;\partial F)$ denotes the connected component which contains the identity, so that there is a canonical monomorphism of topological groups, ${\mathrm{Diff}^\circ}_e(F;\partial F) \to \mathrm{Diff}^\circ(F;\partial F)$.

[2] A sequence of pointed spaces $F \to E \to B$ is a homotopy fibre sequence if F is weakly equivalent to the homotopy fibre of $E \to B$. The homotopy fibre can be defined explicitly as the fibre product $E \times_B PB$, where $PB \to B$ is the path space fibration over B.

Definition 2.1. — *For g, b non-negative integers, the mapping class group $\Gamma_{g,b}$ is the discrete group of path components, $\Gamma_{g,b} := \pi_0(\mathrm{Diff}^\circ(F_{g,b}; \partial F_{g,b}))$.*

Earle and Eells [2] proved that the topological group $\mathrm{Diff}^\circ{}_e(F; \partial F)$ is contractible, for F a smooth, compact, oriented surface of genus $g \geq 2$.

Corollary 2.2. — *For $g \geq 2$ an integer, there is a homotopy equivalence $B\Gamma_{g,b} \simeq B\mathrm{Diff}^\circ(F_{g,b}; \partial F_{g,b})$. In particular, the classifying space $B\Gamma_{g,b}$ classifies isomorphism classes of oriented $F_{g,b}$-bundles.*

There is a model for the classifying space $B\Gamma_{g,b}$ constructed from Teichmüller space, for strictly positive b. Let $\mathcal{H}(F)$ denote the space of hyperbolic metrics on the surface F with geodesic boundary such that each boundary circle has unit length. The hyperbolic model for the moduli space of Riemann surfaces of topological type F is given by

$$\mathcal{M}(F) := \mathcal{H}(F)/\mathrm{Diff}^\circ(F; \partial F).$$

Teichmüller space is defined as the quotient $\mathcal{T}(F) := \mathcal{H}(F)/\mathrm{Diff}^\circ{}_e(F; \partial F)$.

Theorem 2.3. — [2, 3] *Let $F := F_{g,b}$ be a smooth, compact, oriented surface of genus $g > 1$, with b boundary components. The following statements hold.*

1. *The space $\mathcal{H}(F)$ is contractible.*
2. *The space $\mathcal{T}(F)$ is contractible and homeomorphic to $\mathbb{R}^{6g-6+2b}$.*
3. *If $b > 0$, the action of $\Gamma_{g,b}$ on Teichmüller space $\mathcal{T}(F)$ is free and $B\Gamma_{g,b} \simeq \mathcal{M}(F)$.*
4. *If $b = 0$, the action of Γ_g on Teichmüller space $\mathcal{T}(F_{g,0})$ has finite isotropy groups, hence there is a rational homotopy equivalence $B\Gamma_g \simeq_{\mathbb{Q}} \mathcal{M}(F_{g,0})$.*

In particular, the above establishes the relation between the moduli space of Riemann surfaces $\mathcal{M}(F)$ and the mapping class group.

2.2. Stabilization

There are two basic gluing constructions which allow stabilization. Recall that $F_{1,2}$ is diffeomorphic to a torus with the interiors of two disjoint disks removed and $F_{0,1}$ is diffeomorphic to a disk; gluings of smooth manifolds provide concatenation diffeomorphisms of oriented surfaces $F_{g+1,b} \cong F_{g,b} \cup_{S^1} F_{1,2}$ and $F_{g,b} \cong F_{g,b+1} \cup_{S^1} F_{0,1}$.

There are induced natural morphisms of groups $\Gamma_{g,b} \to \Gamma_{g+1,b}$, $\Gamma_{g,b+1} \to \Gamma_{g,b}$ which induce morphisms between the integral cohomologies of the respective classifying spaces. The stability theorems of Harer and Ivanov imply that these are isomorphisms in a stable range.

Theorem 2.4. — [7, 9] *The natural morphisms induce isomorphisms in cohomology $H^*(B\Gamma_{g,b}) \cong H^*(B\Gamma_{g+1,b}) \cong H^*(B\Gamma_{g+1,b+1})$, for $* < g/2 - 1$.*

Let $\Gamma_{\infty,b}$ denote the colimit of the direct system of groups $\Gamma_{*,b}$.

Corollary 2.5. — *The group $H^*(B\Gamma_{\infty,b})$ is isomorphic to the inverse limit of the system $H^*(B\Gamma_{*,b})$ and is independent of b.*

2.3. Infinite loop space structure

The classifying space $B\Gamma_{g,b}$ has perfect fundamental group for $g > 1$ (see [6] for the case $g \geq 2$). Hence, the Quillen plus construction $B\Gamma_{g,b}^{+}$ is defined, with trivial fundamental group, and there is a canonical morphism $B\Gamma_{g,b} \to B\Gamma_{g,b}^{+}$ which induces an isomorphism in homology.

The following theorem motivated the generalized Mumford conjecture [11].

THEOREM 2.6. — [22] *The space $\mathbb{Z} \times B\Gamma_{\infty,b}^{+}$ has the structure of an infinite loop space*(3)*.*

3. GENERALIZED BUNDLE THEORY

3.1. Submersions, fibrewise tangent bundles and jet bundles

A smooth map $\pi : E \to X$ between smooth manifolds is a submersion if the morphism of tangent bundles $TE \to TX$ is surjective on fibres. The vertical tangent bundle $T^{\pi}E$ of the submersion is a vector bundle on E and there is a short exact sequence $T^{\pi}E \to TE \to \pi^{*}TX$ of vector bundles on E.

The fibres of a smooth submersion are smooth manifolds of codimension equal to the dimension of the base. Ehresmann's fibration lemma (see [1] for example) states:

THEOREM 3.1. — *Let $\pi : E \to X$ be a smooth submersion which is proper, then π is a smooth fibre bundle of manifolds.*

The following submersion theorem of Phillips' for open manifolds is used.

THEOREM 3.2. — [18, Theorem A] *Let M, W be smooth manifolds such that M is open. If there exists a smooth surjection of tangent bundles $TM \twoheadrightarrow TX$, then there exists a smooth submersion $M \to X$. Moreover, any two smooth submersions with differentials homotopic through vector vector bundle surjections are homotopic through submersions.*

For $\pi : E \to X$ a smooth submersion, let $J^{k}_{\pi}(E, \mathbb{R})$ denote the fibrewise k-jet bundle; this is a smooth vector bundle on E which is a quotient of the k-jet bundle $J^{k}(E, \mathbb{R})$ of which the fibre at z identifies with $J^{k}(E_{\pi(z)}, \mathbb{R})$, the k-jet bundle of the fibre.

The definition implies that, for k a non-negative integer, there exists a natural surjection $J^{k+1}_{\pi}(E, \mathbb{R}) \to J^{k}_{\pi}(E, \mathbb{R})$ of vector bundles on E. The bundle $J^{0}_{\pi}(E, \mathbb{R})$ is canonically isomorphic to the trivial bundle of rank one on E, which implies the first statement of the following Lemma.

(3)A pointed topological space $(X, *)$ has an infinite loop space structure if there exists a sequence of pointed spaces $(X_n, *)$, and weak homotopy equivalences $X_n \simeq \Omega X_{n+1}$, for $n \geq 0$, such that $(X, *) = (X_0, *)$. (For technical reasons, suppose all the spaces are CW complexes).

(3)(for the purposes of this exposition, suppose without boundary)

Lemma 3.3. — *Let $\pi : E \to X$ be a smooth submersion and let k be a non-negative integer, then the following statements hold.*

1. *A smooth section $\hat{f}$ of $J^k_\pi(E,\mathbb{R})$ induces a smooth function $f : E \to \mathbb{R}$.*
2. *A smooth morphism $f : E \to \mathbb{R}$ induces a smooth section $j^k_\pi(f)$ of the bundle $J^k_\pi(E,\mathbb{R})$.*

There is a canonical identification of the fibrewise 1-jet bundle $J^1_\pi(E,\mathbb{R})$ with $J^0(E,\mathbb{R}) \oplus T^\pi E^*$, where $T^\pi E^*$ is the fibrewise cotangent bundle. In particular, there is a canonical surjection $J^2_\pi(E,\mathbb{R}) \to T^\pi E^*$, corresponding to the linear part of the fibrewise jet bundle. A section of $J^2_\pi(E,\mathbb{R})$ with vanishing linear part has well-defined quadratic part, which corresponds to a quadratic form on $T^\pi E$, by choice of connection.

Definition 3.4. — *Let $\hat{f}$ be a smooth section of the fibrewise 2-jet bundle $J^2_\pi(E,\mathbb{R})$, where $\pi : E \to X$ is a smooth submersion.*

1. *The section $\hat{f}$ is fibrewise non-singular if the linear part is a non-vanishing section of $T^\pi E^*$.*
2. *The section $\hat{f}$ is fibrewise Morse if it has non-degenerate quadratic part whenever the linear part vanishes.*

The submanifold of singular jets in $J^2_\pi(E,\mathbb{R})$ is written $\Sigma_\pi(E,\mathbb{R})$.

Definition 3.5. — *Let $\pi : E \to X$ be a smooth submersion and let $\hat{f}$ be a smooth section of the fibrewise 2-jet bundle. The fibrewise singularity set $\Sigma(\pi,\hat{f}) \subset E$ is the inverse image $\hat{f}^{-1}\Sigma_\pi(E,\mathbb{R})$ of the submanifold of singular jets.*

If $\hat{f}$ identifies with the fibrewise 2-jet prolongation $j^2_\pi(f)$ of a smooth function f, the fibrewise singularity set $\Sigma(\pi,\hat{f})$ is written $\Sigma(\pi,f)$.

A smooth section of the fibrewise 2-jet bundle which is fibrewise Morse is equivalent to a smooth section of the fibrewise cotangent bundle which is transverse to the zero section; this implies the following result.

Lemma 3.6. — [12] *Let $\pi : E \to X$ be a smooth submersion and let $\hat{f}$ be a smooth section of the fibrewise 2-jet bundle, which is fibrewise Morse. Then $\Sigma(\pi,\hat{f})$ is a smooth submanifold of E and the morphism π restricts to a local diffeomorphism*[4] $\pi|_{\Sigma(\pi,\hat{f})} : \Sigma(\pi,\hat{f}) \to X$.

[4] or étale map

3.2. The sheaves $h\mathcal{V}$, $h\mathcal{W}$, $h\mathcal{W}_{\mathrm{loc}}$

Let $\mathfrak{X}$ denote the category of smooth manifolds, without boundary and with a countable basis; all sheaves considered in this section are sheaves of sets on $\mathfrak{X}$.

The lower row of the diagram of sheaves is defined in terms of analytic data.

Definition 3.7. —

1. *Let $h\mathcal{W}$ denote the sheaf which has sections over X the set of pairs $(\pi, \hat{f})$, where $\pi : E \to X$ is a smooth submersion of fibre dimension three, with oriented fibrewise tangent bundle. The morphism $\hat{f}$ is a smooth section of the fibrewise 2-jet bundle $J^2_\pi(E, \mathbb{R})$ such that the following conditions are satisfied:*
   (a) *the morphism $(\pi, f) : E \to X \times \mathbb{R}$ is proper, where f denotes the underlying smooth morphism of $\hat{f}$;*
   (b) *the section $\hat{f}$ is fibrewise Morse.*
2. *Let $h\mathcal{V}$ denote the subsheaf of $h\mathcal{W}$ with sections over X given by pairs $(\pi, \hat{f})$ for which the section $\hat{f}$ is fibrewise non-singular.*

There is a variant of the sheaf $h\mathcal{W}$, in which the properness hypothesis is weakened.

Definition 3.8. — *Let $h\mathcal{W}_{\mathrm{loc}}$ denote the sheaf which has sections over X the set of pairs $(\pi, \hat{f})$, where $\pi : E \to X$ is a smooth submersion of fibre dimension three with oriented fibrewise tangent bundle. The morphism $\hat{f}$ is a smooth section of the fibrewise 2-jet bundle $J^2_\pi(E, \mathbb{R})$ such that the following conditions are satisfied:*

1. *the morphism $(\pi, f) : E \to X \times \mathbb{R}$ restricts to a proper morphism $\Sigma(\pi, \hat{f}) \to X \times \mathbb{R}$;*
2. *the section $\hat{f}$ is fibrewise Morse.*

By construction, there are canonical morphisms of sheaves $h\mathcal{V} \to h\mathcal{W} \to h\mathcal{W}_{\mathrm{loc}}$.

3.3. The sheaves $\mathcal{V}$, $\mathcal{W}$, $\mathcal{W}_{\mathrm{loc}}$

The upper row of diagram (1) is obtained by imposing the condition that the smooth sections are integrable.

Definition 3.9. —

1. *Let $\mathcal{W}$ denote the sheaf with sections over X the set of pairs (π, f) such that $\pi : E \to X$ is a smooth submersion of fibre dimension three, $f : E \to \mathbb{R}$ is a smooth morphism and $(\pi, j^2_\pi f)$ belongs to $h\mathcal{W}(X)$.*
2. *Let $\mathcal{V}$ denote the subsheaf of $\mathcal{W}$ which is defined by the cartesian square of sheaves:*

$$\begin{array}{ccc} \mathcal{V} & \longrightarrow & \mathcal{W} \\ \downarrow & & \downarrow \\ h\mathcal{V} & \longrightarrow & h\mathcal{W}. \end{array}$$

3. *Let $\mathcal{W}_{\mathrm{loc}}$ denote the sheaf with sections over X the set of pairs (π, f), where $\pi : E \to X$ is a smooth submersion of fibre dimension three and $f : E \to \mathbb{R}$ is a smooth morphism such that $(\pi, j^2_\pi f)$ is an element of $h\mathcal{W}_{\mathrm{loc}}$.*

Explicit examples of sections of $\mathcal{W}$ are given by the constructions of Section 6. The definitions yield diagram (1) in which the vertical morphisms are induced by fibrewise 2-jet prolongation.

4. BORDISM

This section addresses the identification of the sequence of representing spaces $|h\mathcal{V}| \to |h\mathcal{W}| \to |h\mathcal{W}_{\mathrm{loc}}|$ by using bordism-theoretic arguments(5).

4.1. Oriented bundles with Morse-like functions

DEFINITION 4.1. — *Let d, n be non-negative integers.*

1. *Let $\mathcal{G}(d,n)$ be the space of oriented d-dimensional subspaces of $\mathbb{R}^{d+n}$.*

   The space $\mathcal{G}(d,n)$ is a classifying space for oriented d-dimensional vector bundles with a morphism of the total space to $\mathbb{R}^{d+n}$ which restricts to a linear embedding on each fibre.
2. *Let $\mathcal{GW}(d,n)$ be the space of triples (V,l,q), where V is an object of $\mathcal{G}(d,n)$, $l : V \to \mathbb{R}$ is a linear map and $q : V \to \mathbb{R}$ is a quadratic form, which is non-degenerate if $l = 0$.*

   The space $\mathcal{GW}(d,n)$ is a classifying space for oriented d-dimensional vector bundles equipped with the additional structure:
   (a) *a morphism of the total space of the bundle to $\mathbb{R}^{d+n}$, which is a linear embedding on each fibre;*
   (b) *a morphism from the total space to $\mathbb{R}$ which restricts to a Morse-like function on each fibre; namely, on each fibre, the morphism has the form $l+q : V \to \mathbb{R}$, where l is linear and q is a quadratic form, which is non-degenerate if l is zero.*
3. *Let $\Sigma(d,n)$ be the subspace of $\mathcal{GW}(d,n)$ which is given by triples of the form $(V,0,q)$, where q is a non-degenerate quadratic form.*
4. *Let $\mathcal{GV}(d,n)$ be the complement $\mathcal{GW}(d,n)\backslash\Sigma(d,n)$.*

There is a diagram of inclusions

$$\mathcal{GV}(d,n) \hookrightarrow \mathcal{GW}(d,n) \hookleftarrow \Sigma(d,n). \tag{2}$$

Remark 4.2. —

1. There is a map $\mathcal{GW}(d,n) \to \mathcal{G}(d,n)$ which forgets the Morse-like function. In particular, (2) is a diagram of spaces over $\mathcal{G}(d,n)$.
2. There is an inclusion $\mathcal{G}(d,n) \hookrightarrow \mathcal{GV}(d+1,n)$, which is induced by $-\oplus\mathbb{R}$ via $V \mapsto (V\oplus\mathbb{R}, l, 0)$, where l is the projection $V\oplus\mathbb{R} \to \mathbb{R}$.

(5) A résumé of the classical Pontrjagin-Thom correspondence between bordism and generalized homology theories is given in [10, Chapter 1]; the influential paper of Quillen, [19], provides the treatment of the cohomological theory.

3. A monomorphism $\mathbb{R}^{d+n} \hookrightarrow \mathbb{R}^{d+n+1}$ induces a natural morphism $\mathcal{G}(d,n) \to \mathcal{G}(d,n+1)$, together with compatible natural morphisms for $\mathcal{GV}$, $\mathcal{GW}$, Σ.

There are tautological oriented d-dimensional bundles T_n on $\mathcal{G}(d,n)$ and U_n on $\mathcal{GW}(d,n)$. Moreover, there is a canonical monomorphism $U_n \hookrightarrow \mathcal{GW}(d,n) \times \mathbb{R}^{d+n}$ into a trivial bundle, with complement $U_n^\perp$ of dimension n; the analogous statement holds for T_n.

Lemma 4.3. —

1. *The injection $\mathcal{G}(d,n) \to \mathcal{GV}(d+1,n)$ is covered by a fibrewise isomorphism of vector bundles $T_n^\perp \to U_n^\perp|_{\mathcal{GV}(d+1,n)}$.*
2. *The normal bundle of the embedding $\Sigma(d,n) \hookrightarrow \mathcal{GW}(d,n)$ is isomorphic to the dual bundle $U_n^*|_{\Sigma(d,n)}$.*
3. *The non-degenerate quadratic form, q, induces an isomorphism between $U_n^*|_{\Sigma(d,n)}$ and $U_n|_{\Sigma(d,n)}$.*

4.2. Thom space constructions

The Thom space $\mathrm{Th}(\xi)$ of a vector bundle ξ is the quotient space $\mathrm{Th}(\xi) := D(\xi)/S(\xi)$, where the pair $(D(\xi), S(\xi))$ corresponds to the disc and sphere bundles associated to ξ.[(6)]

For $M \hookrightarrow \mathbb{R}^N$ an embedding with normal bundle ν such that the total space $E(\nu)$ embeds as a tubular neighbourhood, collapsing the complement of the tubular neighbourhood induces the Pontrjagin-Thom map $S^N \to \mathrm{Th}(\nu)$. This construction generalizes to give the following.

Lemma 4.4. — *Let $Y \hookrightarrow X$ be an immersion of smooth manifolds of codimension d, with normal bundle ν, and let ξ be a vector bundle on X. Then the sequence of spaces $\mathrm{Th}(\xi|(X-Y)) \to \mathrm{Th}(\xi) \to \mathrm{Th}(\xi|Y \oplus \nu)$ is a homotopy cofibre sequence*[(7)].

This applies to the embedding $\Sigma(3,n) \hookrightarrow \mathcal{GW}(3,n)$ and the vector bundle $U_n^\perp$ on $\mathcal{GW}(3,n)$; together with Lemma 4.3, this implies:

Lemma 4.5. — *For n a non-negative integer, there is a homotopy cofibre sequence*

$$\mathrm{Th}(U_n^\perp|\mathcal{GV}(3,n)) \to \mathrm{Th}(U_n^\perp|\mathcal{GW}(3,n)) \to \mathrm{Th}(U_n^\perp \oplus U_n^*|\Sigma(3,n)).$$

The Thom space $\mathrm{Th}(\xi \oplus \theta^1)$ is homeomorphic to the suspension $\Sigma\mathrm{Th}(\xi)$, where θ denotes the trivial bundle of rank one. In particular, a morphism of vector bundles

(6) If the base space is compact, the Thom space is homeomorphic to the one point compactification of the total space.

(7) A sequence $A \to B \to C$ of pointed topological spaces is a homotopy cofibre sequence if C is homotopy equivalent to the mapping cone of $A \to B$.

$\xi \oplus \theta^1 \to \zeta$ induces a morphism $\mathrm{Th}(\xi) \to \Omega\mathrm{Th}(\zeta)$, by adjunction. Such morphisms induce the direct systems which define the infinite loop spaces below[8]:

$$\begin{aligned}
\Omega^\infty \mathbf{h}\mathcal{W} &:= \mathrm{colim}_n \Omega^{2+n}\mathrm{Th}(U_n^\perp|_{\mathcal{GW}(3,n)});\\
\Omega^\infty \mathbf{h}\mathcal{V} &:= \mathrm{colim}_n \Omega^{2+n}\mathrm{Th}(U_n^\perp|_{\mathcal{GV}(3,n)});\\
\Omega^\infty \mathbf{h}\mathcal{W}_{\mathrm{loc}} &:= \mathrm{colim}_n \Omega^{2+n}\mathrm{Th}(U_n^\perp \oplus U_n^*|_{\Sigma(3,n)}).
\end{aligned}$$

Lemma 4.5 implies the following result, which corresponds to the fact that the infinite loop space functor Ω^∞ sends stable homotopy cofibre sequences to homotopy fibre sequences of spaces.

Proposition 4.6. — *There is a homotopy fibre sequence of infinite loop spaces*

$$\Omega^\infty \mathbf{h}\mathcal{V} \to \Omega^\infty \mathbf{h}\mathcal{W} \to \Omega^\infty \mathbf{h}\mathcal{W}_{\mathrm{loc}}.$$

4.3. The morphism $\tau : |h\mathcal{W}| \to \Omega^\infty \mathbf{h}\mathcal{W}$

The construction of a morphism $\tau : |h\mathcal{W}| \to \Omega^\infty \mathbf{h}\mathcal{W}$ uses auxiliary sheaves $h\mathcal{W}^{(r)}$ and $\mathcal{Z}^{(r)}$ which are defined as follows.

Definition 4.7. — *For X a smooth closed manifold and r a non-negative integer:*

1. *let $h\mathcal{W}^{(r)}(X)$ denote the set of sections of $h\mathcal{W}(X)$ with the additional structure: a smooth embedding $\omega : E \to X \times \mathbb{R} \times \mathbb{R}^{2+r}$ over $X \times \mathbb{R}$ and a vertical tubular neighbourhood N;*
2. *let $\mathcal{Z}^{(r)}(X)$ denote the set of maps $X \times \mathbb{R} \to \Omega^{2+r}\mathrm{Th}(U_r^\perp)$.*

The Pontrjagin-Thom construction establishes the following result.

Lemma 4.8. — *There are morphisms of sheaves* $h\mathcal{W} \longleftarrow h\mathcal{W}^{(r)} \longrightarrow \mathcal{Z}^{(r)}$.

The sheaves $h\mathcal{W}^{(r)}$ and $\mathcal{Z}^{(r)}$ form direct systems as r varies, hence there are direct systems of the representing spaces $|h\mathcal{W}^{(r)}|$, $|\mathcal{Z}^{(r)}|$. The first statement of the following Lemma [9] is a consequence of the Whitney embedding theorem.

Lemma 4.9. — *There are weak equivalences:*

1. $\mathrm{hocolim}_r |h\mathcal{W}^{(r)}| \xrightarrow{\simeq} |h\mathcal{W}|$*;*
2. $\mathrm{hocolim}_r |\mathcal{Z}^{(r)}| \xrightarrow{\simeq} \Omega^\infty \mathbf{h}\mathcal{W}$*.*

The Lemma induces a morphism $\tau : |h\mathcal{W}| \to \Omega^\infty \mathbf{h}\mathcal{W}$ in the homotopy category of pointed topological spaces. This restricts to a morphism $\tau_{\mathcal{V}} : |h\mathcal{V}| \to \Omega^\infty \mathbf{h}\mathcal{V}$ and similar constructions define a morphism $\tau_{\mathrm{loc}} : |h\mathcal{W}_{\mathrm{loc}}| \to \Omega^\infty \mathbf{h}\mathcal{W}_{\mathrm{loc}}$ so that the following statement holds:

---

(8) Cf. the construction of Thom spectra, [21].

(9) (in which hocolim denotes the homotopy colimit of the direct system, which is the derived, homotopy-theoretic version of the direct limit)

Lemma 4.10. — *There is a homotopy commutative diagram*

$$\begin{tikzcd} {|h\mathcal{V}|} \arrow[r] \arrow[d, "\tau_{\mathcal{V}}"'] & {|h\mathcal{W}|} \arrow[r] \arrow[d, "\tau"'] & {|h\mathcal{W}_{\mathrm{loc}}|} \arrow[d, "\tau_{\mathrm{loc}}"] \\ \Omega^\infty \mathbf{h}\mathcal{V} \arrow[r] & \Omega^\infty \mathbf{h}\mathcal{W} \arrow[r] & \Omega^\infty \mathbf{h}\mathcal{W}_{\mathrm{loc}}. \end{tikzcd}$$

4.4. Bordism approach to τ

There are bordism descriptions for the cohomology theories represented by the spaces $\Omega^\infty \mathbf{h}\mathcal{W}$, $\Omega^\infty \mathbf{h}\mathcal{V}$ and $\Omega^\infty \mathbf{h}\mathcal{W}_{\mathrm{loc}}$.

For example, consider the case $\Omega^\infty \mathbf{h}\mathcal{W}$; for X a smooth manifold, the set of homotopy classes $[X, \Omega^\infty \mathbf{h}\mathcal{W}]$ identifies with the set of bordism classes of triples $(M, g, \hat{g})$, where M is a closed, smooth manifold of dimension $2 + \dim X$ and g is a morphism, $M \to X \times \mathcal{GW}(3, n)$, such that the projection to X induces a proper morphism, $M \to X$. The morphism $\hat{g}$ is a pull-back of vector bundles of the form

$$\begin{tikzcd} TM \times \mathbb{R} \times \mathbb{R}^j \arrow[r, "\hat{g}"] \arrow[d] & TX \times U_\infty \times \mathbb{R}^j \arrow[d] \\ M \arrow[r, "g"'] & X \times \mathcal{GW}(3, n), \end{tikzcd}$$

for some non-negative integer j. An obstruction theory argument (see [12]) shows that the integer j can be taken to be zero.

Let $(\pi, \hat{f})$ represent an element of $h\mathcal{W}(X)$ then, up to concordance, one may suppose that $f : E \to \mathbb{R}$ is transverse to $0 \in \mathbb{R}$, so that $M := f^{-1}(0)$ is a submanifold of dimension $2 + \dim X$ and the induced morphism $\pi : M \to X$ is proper. The section $\hat{f}$ of the fibrewise 2-jet bundle, restricted to points of M, has the form $\hat{f}_z = l_z + q_z$ satisfying the non-degeneracy hypothesis. There is a classifying map $M \to \mathcal{GW}(3, \infty)$ and it is straightforward to verify that this defines an element of $[X, \Omega^\infty \mathbf{h}\mathcal{W}]$.

Conversely, given a triple $(M, g, \hat{g})$ representing a bordism class , so that $g : M \to X \times \mathcal{GW}(3, \infty)$, set $E := M \times \mathbb{R}$ and let π_E denote the composite morphism $E \to M \to X$. Phillips' submersion theorem (Theorem 3.2) implies that this gives rise to an element of $h\mathcal{W}(X)$.

At the level of concordance classes, this implies the following theorem.

Theorem 4.11. — *The morphism* $\tau : |h\mathcal{W}| \to \Omega^\infty \mathbf{h}\mathcal{W}$ *is a homotopy equivalence.*

Similar considerations apply to show that $\tau_{\mathcal{V}} : |h\mathcal{V}| \to \Omega^\infty \mathbf{h}\mathcal{V}$ and $\tau_{\mathrm{loc}} : |h\mathcal{W}_{\mathrm{loc}}| \to \Omega^\infty \mathbf{h}\mathcal{W}_{\mathrm{loc}}$ are homotopy equivalences.

Corollary 4.12. — *The sequence of spaces,* $|h\mathcal{V}| \to |h\mathcal{W}| \to |h\mathcal{W}_{\mathrm{loc}}|$*, is a homotopy fibre sequence of infinite loop spaces.*

4.5. The morphism $j^2_\pi : \mathcal{W}_{\mathrm{loc}} \to h\mathcal{W}_{\mathrm{loc}}$

The following result is proved in [12].

THEOREM 4.13. — *The morphism of sheaves $j^2_\pi : \mathcal{W}_{\mathrm{loc}} \to h\mathcal{W}_{\mathrm{loc}}$ induces a weak equivalence between the representing spaces.*

The proof is geometric in nature, using a bordism theoretic description of $\mathcal{W}_{\mathrm{loc}}(X)$, together with an application of Phillips' submersion theorem, Theorem 3.2.

4.6. Identification of the space $\Omega^\infty \mathbf{h}\mathcal{V}$ and the morphism $j^2_\pi : \mathcal{V} \to h\mathcal{V}$

The morphism $\mathcal{G}(2,n) \to \mathcal{GW}(3,n)$ gives rise to a morphism of Thom spaces $\mathrm{Th}(T_n^\perp) \to \mathrm{Th}(U_n^\perp|_{\mathcal{GW}(3,n)})$. The presence of the quadratic form does not affect the homotopy type of the associated infinite loop space:

LEMMA 4.14. — *There is a weak equivalence $\Omega^\infty \mathbf{h}\mathcal{V} \simeq \mathrm{colim}_n \Omega^{2+n} \mathrm{Th}(T_n^\perp)$.*

A standard argument involving complexification[10] implies the following result.

PROPOSITION 4.15. — *The space $|h\mathcal{V}|$ is equivalent to the infinite loop space, $\Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}$.*

The morphism $j^2_\pi : \mathcal{V} \to h\mathcal{V}$ induces a map between concordance classes $\mathcal{V}[X] \to h\mathcal{V}[X]$, which is induced by a map $|\mathcal{V}| \to |h\mathcal{V}|$ between the representing spaces. The space $|h\mathcal{V}|$ is equivalent to the space $\Omega^\infty \mathbf{h}\mathcal{V}$ (remark following Theorem 4.11) and hence to $\Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}$ by Proposition 4.15.

The sheaf $\mathcal{V}$ can be interpreted in terms of surface bundles on $X \times \mathbb{R}$, by Ehresmann's fibration lemma. Hence the functor $X \mapsto \mathcal{V}[X]$ is represented by the space $\amalg_F B\mathrm{Diff}^\circ(F, \partial F)$, as F ranges over diffeomorphism classes of smooth, closed oriented surfaces (not necessarily connected). The natural transformation $\mathcal{V}[X] \to h\mathcal{V}[X]$ is represented by a map $\alpha : \amalg_F B\mathrm{Diff}^\circ(F, \partial F) \to \Omega^\infty \mathbf{h}\mathcal{V} \simeq \Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}$, which is related to the morphism α_∞ of the Introduction.

5. APPLICATION OF VASSILIEV'S FIRST MAIN THEOREM

Theorem 1.3 is deduced from Vassiliev's first main theorem; the proof uses techniques from sheaf homotopy theory.

[10] (passage from the structure group SO to U)

5.1. Vassiliev's first main theorem

Definition 5.1. — *Let* $\mathfrak{U} \subset J^2(\mathbb{R}^r, \mathbb{R})$ *denote the space of 2-jets represented by smooth functions* $f : (\mathbb{R}^r, z) \to \mathbb{R}$ *such that* $f(z) = 0$ *and* $df(z) = 0$ *and* $\det(d^2 f(z)) = 0$.

The space $\mathfrak{U}$ corresponds to the space of 2-jets with singularities which have critical value zero and which are not Morse.

Definition 5.2. — *For* $N^r, \partial N^r$ *a smooth, compact manifold with boundary and* $\psi : N \to \mathbb{R}$ *a smooth function such that* $j^2\psi(z) \notin \mathfrak{U}$ *in a neighbourhood of* ∂N*, define the spaces:*

1. $\Phi((N, \partial N), \psi) := \{f \in C^\infty(N, \mathbb{R}) | f \equiv_{\partial N} \psi, j^2 f \notin \mathfrak{U}\}$;
2. $h\Phi((N, \partial N), \psi) := \{\hat{f} \in \Gamma J^2(N, \mathbb{R}) | \hat{f} \equiv_{\partial N} j^2\psi, \hat{f} \notin \mathfrak{U}\}$.

(Here, $\equiv_{\partial N}$ *indicates equality in a neighbourhood of* ∂N*.)*

Jet prolongation defines a map $j^2 : \Phi((N, \partial N), \psi) \to h\Phi((N, \partial N), \psi)$. A special case of Vassiliev's main theorem reads as follows:

Theorem 5.3. — [24, 23] *The map* $j^2 : \Phi((N, \partial N), \psi) \to h\Phi((N, \partial N), \psi)$ *induces an isomorphism in integral homology.*

5.2. Indications on the proof of Theorem 1.3

The Whitehead theorem implies that it is sufficient to show that the morphism $j^2_\pi : |\mathcal{W}| \to |h\mathcal{W}|$ induces an isomorphism on integral homology, using the fact that the spaces $|\mathcal{W}|$, $|h\mathcal{W}|$ are simple(11). This fact is deduced from the existence of compatible monoid structures on the spaces, together with the fact that $|h\mathcal{W}|$ has the structure of an infinite loop space, by Theorem 1.4.

The next step is to extend $\mathcal{W}$, $h\mathcal{W}$ to weakly equivalent sheaves $\mathcal{W}^0$, $h\mathcal{W}^0$ in which the Morse condition is only imposed in a neighbourhood of the critical value $f^{-1}(0)$. There is a canonical extension of the jet-prolongation morphism to a morphism $j^2_\pi : \mathcal{W}^0 \to h\mathcal{W}^0$ and it is sufficient to show that this induces an isomorphism in integral homology.

The sheaves $\mathcal{W}^0$, $h\mathcal{W}^0$ admit homotopy colimit decompositions, expressed in terms of the functor β of Definition A.13. Theorem 1.3 is deduced by applying Proposition A.17: the proof reduces to showing that the morphism j^2_π induces a homology equivalence between fibres over the same point. This follows from Vassiliev's main theorem by identifying the morphism between the fibres explicitly.

(11) A connected space is simple if it has abelian fundamental group which acts trivially on the higher homotopy groups.

6. ELEMENTARY MORSE SINGULARITIES

The proof of Theorem 1.5 requires an analysis of standard models for the elements of $\mathcal{W}$ in terms of certain multi-elementary Morse singularities. The constructions introduced here are exploited in Section 7.

6.1. Morse vector spaces and vector bundles

Definition 6.1. —

1. *A Morse vector space is a pair (V,ρ), where V is a finite dimensional real vector space equipped with an inner product and $\rho : V \to V$ is a linear, isometric involution. The involution ρ induces a decomposition $V \cong V^{\rho} \oplus V^{-\rho}$, $u \mapsto (u_+, u_-)$, where ρ acts trivially on V^{ρ} and by multiplication by -1 on $V^{-\rho}$.*
2. *The Morse index of (V,ρ) is the dimension of $V^{-\rho}$.*
3. *The Morse function of (V,ρ) is the smooth function $f_V : V \to \mathbb{R}$ given by*

$$f_V(u) := \langle u, \rho u\rangle \cong ||u_+||^2 - ||u_-||^2.$$

The Morse vector spaces provide a good local model for elementary Morse functions, namely those with a single critical point.

Definition 6.2. — *For (V,ρ) a Morse vector space, the saddle of (V,ρ) is the smooth manifold with boundary:* $\mathrm{Saddle}(V,\rho) := \{u \in V|\ ||u_+||^2||u_-||^2 \leq 1\}$.

Example 6.3. — Let (V,ρ) be a Morse vector space of dimension three and of Morse index one; the above construction yields $f_V : \mathrm{Saddle}(V,\rho) \to \mathbb{R}$. The function f_V is fibrewise singular; there is an isolated critical point in the fibre above $0 \in \mathbb{R}$ and the morphism f_V restricts to a bundle of closed surfaces above $\mathbb{R}\backslash\{0\}$. The fibre above a point of $(0,\infty)$ is a hyperboloid of two sheets and above $(-\infty,0)$ is a hyperboloid of one sheet; the singular fibre above $\{0\}$ is a cone.

Lemma 6.4. — *There is a smooth embedding of codimension zero:*

$$\begin{array}{rcl} \mathrm{Saddle}(V,\rho)\backslash V^{\rho} & \hookrightarrow & D(V^{\rho}) \times S(V^{-\rho}) \times \mathbb{R} \\ u & \mapsto & (||u_-||u_+, ||u_-||^{-1}u_-, f_V(u)) \end{array}$$

with complement $0 \times S(V^{-\rho}) \times [0,\infty)$. In particular, there is a diffeomorphism of the boundary of $\mathrm{Saddle}(V,\rho)$ *with $S(V^{\rho}) \times S(V^{-\rho}) \times \mathbb{R}$.*

There is an analogous smooth embedding $\mathrm{Saddle}(V,\rho)\backslash V^{-\rho} \hookrightarrow S(V^{\rho}) \times D(V^{-\rho}) \times \mathbb{R}$ with complement $S(V^{\rho}) \times 0 \times (-\infty, 0]$.

Example 6.5. — Let (V,ρ) be a Morse vector space as in Example 6.3 and let M be a smooth, oriented, closed surface with boundary, with an embedding $D^2 \times S^0 \hookrightarrow \mathrm{Int}M$ into the interior of M. There is a smooth 3-manifold constructed by gluing:

$$W(M,(V,\rho)) \cong (M^{\circ} \times \mathbb{R}) \cup_{S^1 \times S^0 \times \mathbb{R}} \mathrm{Saddle}(V,\rho),$$

where M° is obtained by removing the interior of the embedded disks $D^2 \times S^0$, such that f_V extends to a smooth function $\widetilde{f_V} : W(M,(V,\rho)) \to \mathbb{R}$. This defines an element of $\mathcal{W}(*)$.

For example, if $M = S^2$ is the sphere with two disjoint embedded disks then the bundle of surfaces over $\mathbb{R}\backslash\{0\}$ has fibre above $(0,\infty)$ a sphere and, above $(-\infty,0)$, a torus. The singular fibre above $\{0\}$ is topologically a sphere with two disjoint points identified.

The notion of a Morse vector space extends to that of a Morse vector bundle.

Definition 6.6. — *A Morse vector bundle over X is a triple (V,p,ρ), where $p : V \to X$ is a smooth Riemannian vector bundle and $\rho : V \to V$ is a fibrewise linear isometric involution over X.*

The saddle of a Morse vector bundle (V,p,ρ) is the smooth manifold over X obtained by applying the saddle construction fibrewise.

Examples 6.3 and 6.5 generalize to the parametrized situation and give examples of sections of $\mathcal{W}(X)$ not in $\mathcal{V}(X)$.

6.2. Regularization of elementary Morse singularities

There are two standard ways of regularizing the Morse singularity of the function f_V, by removing the embedded subspace V^ρ (respectively $V^{-\rho}$).

Lemma 6.7. — *Let (V,ρ) be a Morse vector space. There exists a proper, regular function:*

$$f_V^+ : \mathrm{Saddle}(V,\rho)\backslash V^\rho \to \mathbb{R}$$

such that f_V^+ agrees with f_V on an open subset of $\mathrm{Saddle}(V,\rho)$ *which contains the boundary and on the subset* $\{u \in \mathrm{Saddle}(V,\rho) | f_V(u) \leq -1\}$.

There exists an analogous construction of a proper, regular function: $f_V^- : \mathrm{Saddle}(V,\rho)\backslash V^{-\rho} \to \mathbb{R}$.

Remark 6.8. — The hypothesis on the open subset of agreement of f_V^+ is necessary for two reasons: to ensure that smooth gluing is possible (using the neighbourhood of the boundary) and to ensure that there is an explicit form of the restriction of f_V^+ to the submanifold $f_V(u) = -1$.

Lemma 6.9. — *The functions f_V^+, f_V^- induce diffeomorphisms*

$$\begin{aligned} \mathrm{Saddle}(V,\rho)\backslash V^\rho &\hookrightarrow D(V^\rho) \times S(V^{-\rho}) \times \mathbb{R}, \\ \mathrm{Saddle}(V,\rho)\backslash V^{-\rho} &\hookrightarrow S(V^\rho) \times D(V^{-\rho}) \times \mathbb{R}, \end{aligned}$$

by $u \mapsto (||u_-||u_+, ||u_-||^{-1}u_-, f_V^+(u))$ and $u \mapsto (||u_+||^{-1}u_+, ||u_+||u_-, f_V^-(u))$ respectively.

The following Lemma ensures that elementary Morse functions are modelled by saddles; moreover, there are standard ways to reparametrize Morse functions upon saddles.

Lemma 6.10. — [12] *Let N be a smooth manifold equipped with an elementary Morse function $f : N \to \mathbb{R}$, with unique critical value 0, then there exists a Morse vector space (V, ρ) and a codimension zero embedding $\lambda : \mathrm{Saddle}(V, \rho) \to N \backslash \partial N$, such that $f\lambda = f_V$.*

Definition 6.11. — *For N, f as above, the positive regularization is the pair $(N_+^{\mathrm{rg}}, f_+^{\mathrm{rg}})$, where $N_+^{\mathrm{rg}} := N \backslash \lambda(V^\rho)$ and f_+^{rg} is given by patching f and f_V^+. If N is oriented, N_+^{rg} is given the induced orientation.*

The negative regularization $(N_-^{\mathrm{rg}}, f_-^{\mathrm{rg}})$ is defined in the analogous way.

Lemma 6.12. — *For $(N_+^{\mathrm{rg}}, f_+^{\mathrm{rg}})$ the positive regularization as above, the function f_+^{rg} is smooth and proper.*

6.3. Surgery and the long trace construction

The following surgery construction is related to Example 6.5; there is an evident parametrized version of the construction.

Definition 6.13. — *Let M be a smooth, compact manifold, equipped with a codimension zero embedding, $e : D(V^\rho) \times S(V^{-\rho}) \to M \backslash \partial M$, for a Morse vector space (V, ρ) with $\dim V = \dim M + 1$.*

The long trace of e is the smooth manifold $\mathrm{trace}(e)$ which is obtained as the pushout of the codimension zero embeddings

$$\begin{tikzcd} \mathrm{Saddle}(V,\rho)\backslash V^\rho \arrow[r, hook] \arrow[d, hook] & \mathrm{Saddle}(V,\rho) \\ (M \times \mathbb{R})\backslash e(0 \times S(V^{-\rho})) \times [0, \infty). & \end{tikzcd}$$

The long trace is equipped with the elementary Morse function which is the smooth height function on the complement of $\mathrm{Saddle}(V, \rho)$ and identifies with the function f_V on the copy of $\mathrm{Saddle}(V, \rho)$.

The regularization constructions of the previous section give two ways in which to regularize the function f_V, which correspond to removing the subspaces V^ρ (respectively $V^{-\rho}$) from the embedded copy of $\mathrm{Saddle}(V, \rho)$.

There is a related surgery construction which corresponds to changing the choice of regularization:

Definition 6.14. — *For $e : D(V^\rho) \times_X S(V^{-\rho}) \hookrightarrow M$ a smooth embedding, where $q : M \to X$ is a smooth bundle of d-manifolds with an orientation of the vertical tangent bundle and (V, ρ) is a Morse vector bundle over X, let $q^\flat : M^\flat \to X$ be the bundle of d-manifolds which is obtained by fibrewise surgery, by removing the interior of $D(V^\rho) \times_X S(V^{-\rho})$ and gluing in $S(V^\rho) \times_X D(V^{-\rho})$.*

Example 6.15. — Let $M = S^2$ be the sphere with an embedding $e : D^2 \times S^0 \hookrightarrow M$; the manifold $M^\flat$ is diffeomorphic to the torus. The construction can be reversed by inverting the rôle of the Morse index.

7. STRATIFICATION OF THE SHEAVES $\mathcal{W}$, $\mathcal{W}_{\mathrm{loc}}$

The proof of the generalized Mumford conjecture requires an analysis of the homotopy fibre of the map of representing spaces, $|\mathcal{W}| \to |\mathcal{W}_{\mathrm{loc}}|$. This involves the formation of homotopy colimit decompositions of the sheaves $\mathcal{W}$, $\mathcal{W}_T$ via stratifications (see Corollary 7.4). The strategy of the construction is resumed in the following sequence of results, which require the introduction of certain auxiliary sheaves.

The first step replaces the sheaves $\mathcal{W}$, $\mathcal{W}_{\mathrm{loc}}$ by sheaves $\mathcal{L}$, $\mathcal{L}_{\mathrm{loc}}$, in which there are standard forms for neighbourhoods of the critical points.

Proposition 7.1. — *There is a commutative diagram of sheaves,*

$$\begin{tikzcd} \mathcal{L} \arrow[r, "\simeq"] \arrow[d] & \mathcal{W}^{\mu} \arrow[r, "\simeq"] \arrow[d] & \mathcal{W} \arrow[d] \\ \mathcal{L}_{\mathrm{loc}} & \mathcal{W}^{\mu}_{\mathrm{loc}} \arrow[l, "\simeq"'] \arrow[r, "\simeq"'] & \mathcal{W}_{\mathrm{loc}}, \end{tikzcd}$$

in which the horizontal morphisms are weak equivalences.

There are compatible stratifications of the sheaves $\mathcal{L}$, $\mathcal{L}_{\mathrm{loc}}$, which give rise to diagrams $\mathcal{L}_T \to \mathcal{L}$, $\mathcal{L}_{\mathrm{loc},T} \to \mathcal{L}_{\mathrm{loc}}$, indexed over the small category $\mathcal{K}^{\mathrm{op}}$. These provide homotopy decompositions of $\mathcal{L}$, $\mathcal{L}_{\mathrm{loc}}$ respectively, by the following result, using the techniques of Appendix A.6, where the definition of the homotopy colimit (as a sheaf) of a diagram of sheaves is given.

Proposition 7.2. — *There is a commutative diagram of sheaves*

$$\begin{tikzcd} \mathrm{hocolim}_{T\in\mathcal{K}}\mathcal{L}_T \arrow[r, "\simeq"] \arrow[d] & \mathcal{L} \arrow[d] \\ \mathrm{hocolim}_{T\in\mathcal{K}}\mathcal{L}_{\mathrm{loc},T} \arrow[r, "\simeq"'] & \mathcal{L}_{\mathrm{loc}}, \end{tikzcd}$$

in which the horizontal morphisms are weak equivalences.

Corollary 7.4 is expressed in terms of diagrams of sheaves $\mathcal{W}_T$, $\mathcal{W}_{\mathrm{loc},T}$, which are indexed over $\mathcal{K}^{\mathrm{op}}$.

Proposition 7.3. — *There are natural commutative diagrams of sheaves, for $T \in \mathcal{K}$:*

$$\begin{tikzcd} \mathcal{L}_T \arrow[r, "\simeq"] \arrow[d] & \mathcal{W}_T \arrow[d] \\ \mathcal{L}_{\mathrm{loc},T} \arrow[r, "\simeq"'] & \mathcal{W}_{\mathrm{loc},T}, \end{tikzcd}$$

in which the horizontal morphisms are weak equivalences.

Propositions 7.1, 7.2, 7.3 and homotopy invariance of homotopy colimits imply the following:

Corollary 7.4. — *The homotopy fibre of the morphism* $|\mathcal{W}| \to |\mathcal{W}_{\mathrm{loc}}|$ *is weakly equivalent to the homotopy fibre of the morphism* $\mathrm{hocolim}_{T\in\mathcal{K}}|\mathcal{W}_T| \to \mathrm{hocolim}_{T\in\mathcal{K}}|\mathcal{W}_{\mathrm{loc},T}|$.

7.1. The sheaves $\mathcal{L}$ and $\mathcal{L}_{\mathrm{loc}}$

The passage to the sheaves $\mathcal{L}$, $\mathcal{L}_{\mathrm{loc}}$ corresponds to requiring standard forms for neighbourhoods of the critical points arising in $\mathcal{W}$ and $\mathcal{W}_{\mathrm{loc}}$.

Definition 7.5. — *Let* $\mathcal{L}_{\mathrm{loc}}$ *denote the sheaf which has sections over* X *the set of triples* (p, g, V)*, where:*

1. p *is an étale map* $Y \to X$*;*
2. g *is a smooth function* $Y \to \mathbb{R}$ *such that the morphism* $(p, g) : Y \to X \times \mathbb{R}$ *is proper;*
3. $\omega : V \to Y$ *is a three-dimensional oriented Morse vector bundle.*

The above definition implies that $Y \to X$ is a finite étale covering.

Definition 7.6. — *Let* $\mathcal{L}$ *denote the sheaf which has sections over* X *the set of tuples* $(p, g, V, \pi, f, \lambda)$ *such that*

1. $(p, g, V) \in \mathcal{L}_{\mathrm{loc}}(X)$*;*
2. $(\pi, f) \in \mathcal{W}(X)$*;*
3. $\lambda : \mathrm{Saddle}(V, \rho) \to E$ *is a smooth embedding such that* $\Sigma(\pi, f) \subset \mathrm{image}(\lambda)$ *and* λ *respects the orientation along the fibrewise singularity set;*
4. f *is a function such that* $f\lambda(u) = f_V(u) + g(\omega(u))$*, so that* $f\lambda$ *has the same fibrewise singularity set as* f_V *and* g *corresponds to the critical value function.*

Tuples as above satisfy $Y = \Sigma(\pi, f)$ and the vector bundle $V \to Y$ is isomorphic to the restriction of $T^\pi E$ to $\Sigma(\pi, f)$.

The definitions yield canonical morphisms of sheaves $\mathcal{W} \longleftarrow \mathcal{L} \longrightarrow \mathcal{L}_{\mathrm{loc}}$. The proof of Proposition 7.1 requires the introduction of the auxiliary sheaves $\mathcal{W}^\mu$ and $\mathcal{W}^\mu_{\mathrm{loc}}$; the relative surjectivity criterion of Proposition A.7 is used to show that the morphisms $\mathcal{L} \to \mathcal{W}^\mu$ and $\mathcal{W}^\mu_{\mathrm{loc}} \to \mathcal{L}_{\mathrm{loc}}$ are weak equivalences. The remaining comparisons are straightforward.

7.2. The category $\mathcal{K}$

The Morse index of a Morse vector space, (V, ρ), is an integer in the interval $[0, \dim V]$. When considering the regularization of elementary Morse singularities, one can attribute an integer in $\{1, -1\}$ to a critical point according to the way in which the function is regularized, corresponding to sending the critical value either to ∞ or $-\infty$. This motivates the following definition:

Definition 7.7. — *Let* $\mathcal{K}$ *denote the category with objects finite sets over* $[\mathbf{3}] := \{0, 1, 2, 3\}$ *and with morphisms injective maps over* $[\mathbf{3}]$*,* $\iota : S \hookrightarrow T$*, together with a function* $\epsilon : T\backslash\iota S \to \{\pm 1\}$*. The composition of morphisms is given in the obvious way.*

A morphism in $\mathcal{K}$ is the composite of 'elementary' morphisms of the form $S \hookrightarrow S \cup \{a\}$, monomorphism of sets over $[\mathbf{3}]$, together with the value $\epsilon(a) \in \{\pm 1\}$.

7.3. Stratifying - the sheaves $\mathcal{L}_T$ and $\mathcal{L}_{\mathrm{loc},T}$

The sheaves $\mathcal{L}$ and $\mathcal{L}_{\mathrm{loc}}$ are stratified by taking into account the Morse index and the possible regularizations of the fibrewise singularity sets. The following definition is justified by the observation that, for any element (p, g, V) of $\mathcal{L}_{\mathrm{loc}}(X)$, the function g is locally either bounded above or bounded below (the statement is made precise in [12]); this follows from the properness of the morphism (p, g).

Definition 7.8. — *For T an object of $\mathcal{K}$, let $\mathcal{L}_{\mathrm{loc},T}$ denote the sheaf with sections over X the set of tuples (p, g, V, δ, h) where:*

1. *$(p, g, V) \in \mathcal{L}_{\mathrm{loc}}(X)$;*
2. *$\delta : Y \to \{-1, 0, +1\}$ is a continuous function;*
3. *$h : T \times X \to \delta^{-1}(0) \subset Y$ is a diffeomorphism over $[\mathbf{3}] \times X$, where the structure morphism $Y \to [\mathbf{3}]$ is induced by the Morse index;*

such that, for each $x \in X$, there exists a neighbourhood U of x such that g is bounded below on $p^{-1}(U) \cap \delta^{-1}(+1)$ and bounded above on $p^{-1}(U) \cap \delta^{-1}(-1)$.

Remark 7.9. —

1. The function δ is locally constant; the existence of h implies that δ is constant on each sheet of $\delta^{-1}(0)$.
2. The data h, δ are introduced since a choice of regularization is allowed only where the function g is locally bounded.

Lemma 7.10. — *The association $T \mapsto \mathcal{L}_{\mathrm{loc},T}$ is contravariantly functorial, where for a generating morphism $S \hookrightarrow S \cup \{a\}$, $(p, g, V, \delta, h) \in \mathcal{L}_{\mathrm{loc},S\cup\{a\}}(X)$ has image (p, g, V, δ', h'), where the subset $(\delta')^{-1}(0)$ corresponds to $S \times X \hookrightarrow (S \cup \{a\})(X)$ via h, the morphism h' is the induced morphism and the value of δ' on $\{a\} \times X$ is $\epsilon(a)$.*

Definition 7.11. — *For T an object of $\mathcal{K}$,*

1. *let $\mathcal{L}'_T$ denote the pullback of the diagram $\mathcal{L} \to \mathcal{L}_{\mathrm{loc}} \leftarrow \mathcal{L}_{\mathrm{loc},T}$;*
2. *let $\mathcal{L}_T$ denote the subsheaf of $\mathcal{L}'_T$ which is given by elements for which g is identically zero on $\delta^{-1}(0)$.*

These constructions yield the diagram of morphisms of sheaves

$$\begin{tikzcd} \mathcal{L}_T \arrow[r] \arrow[d] & \mathcal{L} \arrow[d] \\ \mathcal{L}_{\mathrm{loc},T} \arrow[r] & \mathcal{L}_{\mathrm{loc}}, \end{tikzcd}$$

natural in T.

Proposition 7.2 asserts that the morphisms $\mathrm{hocolim}_{T\in\mathcal{K}}\mathcal{L}_T \to \mathcal{L}$ and $\mathrm{hocolim}_{T\in\mathcal{K}}\mathcal{L}_{\mathrm{loc},T} \to \mathcal{L}_{\mathrm{loc}}$ are weak equivalences. The sheaf theoretic model for the homotopy colimit introduced in Definition A.13 is used, together with certain auxiliary sheaves; the proofs appeal to the relative surjectivity criterion of Proposition A.7. The reader is referred to [12] for the details.

7.4. The sheaves $\mathcal{W}_T$, $\mathcal{W}_{\mathrm{loc},T}$

The sheaves $\mathcal{W}_T$, $\mathcal{W}_{\mathrm{loc},T}$ introduced in this section play a fundamental rôle in the proof of the generalized Mumford conjecture.

Definition 7.12. — *For T an object of $\mathcal{K}$, let $\mathcal{W}_{\mathrm{loc},T}$ denote the sheaf with sections over X the set of three-dimensional smooth, oriented Riemannian vector bundles, $\omega : V \to T \times X$, equipped with a fibrewise linear isometric involution $\rho : V \to V$ such that the fibres above $t \times X \subset T \times X$ have Morse index given by the image of t in $[\mathbf{3}]$. The sheaf structure is given by the pull-back of vector bundles.*

Lemma 7.13. — *The association $T \mapsto \mathcal{W}_{\mathrm{loc},T}$ is contravariantly functorial in $\mathcal{K}$.*

Definition 7.14. — *For T an object of $\mathcal{K}$, let $\mathcal{W}_T$ denote the sheaf with sections over X the set of data $((V,\rho), q : M \to X, e)$ satisfying*

1. *(V,ρ) is an element of $\mathcal{W}_{\mathrm{loc},T}$;*
2. *$q : M \to X$ is a smooth bundle of closed surfaces, with oriented vertical tangent bundle;*
3. *$e : D(V^\rho) \times_{T\times X} S(V^{-\rho}) \to M$ is a smooth embedding over X, respecting the vertical orientations.*

The functoriality of $\mathcal{W}_{-}$ with respect to $\mathcal{K}$ corresponds to the alternative choices of regularizations. This uses the surgery construction of Definition 6.14.

Lemma 7.15. — *The association $T \mapsto \mathcal{W}_T$ is contravariantly functorial in $\mathcal{K}$, with respect to the following structure for elementary morphisms in $\mathcal{K}$ of the form $S \hookrightarrow S \cup \{a\}$. An element $((V,\rho),q,e)$ maps to $((V',\rho'),q',e')$, where*

1. *$(M',q') := (M^\flat, q^\flat)$ if $\epsilon(a) = -1$, and $(M',q') = (M,q)$ otherwise;*
2. *(V',ρ') is given by functoriality of $\mathcal{W}_{\mathrm{loc},T}$;*
3. *e' is the induced embedding.*

By construction, there is a forgetful map $\mathcal{W}_T \to \mathcal{W}_{\mathrm{loc},T}$, which is natural in T.

7.5. Relating $\mathcal{L}_{\mathrm{loc},T}$ and $\mathcal{W}_{\mathrm{loc},T}$

There is a morphism of sheaves $\mathcal{L}_{\mathrm{loc},T} \to \mathcal{W}_{\mathrm{loc},T}$ which is given by $(p,g,V,\delta,h) \mapsto h^*(V)$. Similarly, there is a map $\mathcal{W}_{\mathrm{loc},T} \to \mathcal{L}_{\mathrm{loc},T}$ which is induced by taking δ and g to be identically zero.

The relative surjectivity criterion of Proposition A.7 is used to show that the morphism $\mathcal{W}_{\mathrm{loc},T} \to \mathcal{L}_{\mathrm{loc},T}$ is a weak equivalence; it is straightforward to deduce that $\mathcal{L}_{\mathrm{loc},T} \to \mathcal{W}_{\mathrm{loc},T}$ is a weak equivalence.

7.6. Regularization and the natural transformation $\mathcal{L}_T \to \mathcal{W}_T$

A natural morphism of sheaves $\mathcal{L}_T \to \mathcal{W}_T$ is constructed by using the regularization construction below.

DEFINITION 7.16. — *For $(\pi, f, p, g, V, \delta, h, \lambda)$ representing a section of $\mathcal{L}_T(X)$, write V_+ for $V|_{\delta^{-1}(1)}$, V_- for $V|_{\delta^{-1}(-1)}$ and V_0 for $V|_{\delta^{-1}(0)}$.*

Let $(E^{\mathrm{rg}}, \pi^{\mathrm{rg}}, f^{\mathrm{rg}})$ denote the structure:

1. *$E^{\mathrm{rg}} := E\backslash\lambda(V_+^{\rho} \cup V_0^{\rho} \cup V_-^{-\rho})$;*
2. *$\pi^{\mathrm{rg}} := \mu|_{E^{\mathrm{rg}}}$;*
3. *$f^{\mathrm{rg}} : E^{\mathrm{rg}} \to \mathbb{R}$ the extension of $f|_{E\backslash \mathrm{Im}(\lambda)}$ by f_V^+ on the image of $V_+ \cup V_0$ and f_V^- on the image of V_-.*

LEMMA 7.17. — *Let $(\pi, f, p, g, V, \delta, h, \lambda)$ represent a section of $\mathcal{L}_T(X)$, then E^{rg} is an open subset of E, $\pi^{\mathrm{rg}} : E^{\mathrm{rg}} \to X$ is a smooth submersion and f^{rg} is regular on each fibre of π^{rg}. Moreover, $(\pi^{\mathrm{rg}}, f^{\mathrm{rg}}) : E^{\mathrm{rg}} \to X \times \mathbb{R}$ is a smooth, proper submersion.*

Ehresmann's fibration lemma implies that the map $(\pi^{\mathrm{rg}}, f^{\mathrm{rg}}) : E^{\mathrm{rg}} \to X \times \mathbb{R}$ is a smooth bundle of closed, oriented surfaces.

DEFINITION 7.18. — *For $(\pi, f, p, g, V, \delta, h, \lambda)$ representing a section of $\mathcal{L}_T(X)$, let $((V, \rho), q : M \to X, e)$ denote the element of $\mathcal{W}_T(X)$ defined by:*

1. *$q : M \to X$ is the fibre of E^{rg} above $X \times \{-1\}$;*
2. *$e : \{v \in \mathrm{Saddle}(V_0, \rho) | f_V(v) = -1\} \to M$ is the induced embedding, which has source $D(V_0^{\rho}) \times_{T\times X} S(V_0^{-\rho})$;*
3. *(V, ρ) is the object $h^*(V_0)$, considered as an object of $\mathcal{W}_{\mathrm{loc},T}(X)$.*

PROPOSITION 7.19. — *The construction of Definition 7.18 defines a morphism of sheaves $\mathcal{L}_T \to \mathcal{W}_T$, which is natural in T.*

Proof. — The proof of the naturality with respect to T is a verification that the functorial behaviour of $\mathcal{W}_T$ defined in Lemma 7.15 is compatible with the functorial behaviour of $\mathcal{L}_T$. □

The natural morphism fits into a commutative diagram

$$\begin{tikzcd} \mathcal{L}_T \arrow[r] \arrow[d] & \mathcal{W}_T \arrow[d] \\ \mathcal{L}_{\mathrm{loc},T} \arrow[r] & \mathcal{W}_{\mathrm{loc},T}, \end{tikzcd}$$

hence the proof of Proposition 7.3 is completed by showing that the morphism $\mathcal{L}_T \to \mathcal{W}_T$ is a weak equivalence.

The long trace construction (Definition 6.13) induces a morphism (not natural with respect to T) of sheaves, $\mathcal{W}_T \to \mathcal{L}_T$. Proposition 7.3 can be deduced from the following result (for the proof, the reader is referred to [12]).

PROPOSITION 7.20. — *The morphism $\mathcal{W}_T \to \mathcal{L}_T$ is a weak equivalence.*

8. REFINEMENTS OF THE HOMOTOPY COLIMIT DECOMPOSITIONS

The stratifications of the previous section require to be refined in two ways. Firstly, it is necessary to add controlled boundaries to the 3-manifolds which are considered, so as to allow concatenation. The second refinement reduces to the case in which the fibres of the bundles are connected; this condition is required in Lemma 9.1.

8.1. Controlled boundaries

Let C be a non-empty closed, smooth, oriented 1-manifold. Any such manifold is null-bordant: there exists a smooth, compact oriented 2-manifold K with boundary $\partial K = C$ as oriented 1-manifolds.

Example 8.1. — For the application to the stable moduli space of surfaces, take C to be the manifold $S^1 \amalg -S^1$.

All of the sheaves considered in the previous section can be modified by allowing a constant boundary derived from C. For example, consider the sheaf $\mathcal{W}$.

Definition 8.2. — *Let ${}^\partial\mathcal{W}$ denote the sheaf with sections over X the set of pairs (π, f), where $\pi : E \to X$ is a smooth submersion of fibre dimension three with oriented fibrewise tangent bundle, $f : E \to \mathbb{R}$ is a smooth map and ∂E over $X \times \mathbb{R}$ has a neighbourhood which is diffeomorphic to a collar on $X \times C \times \mathbb{R}$, respecting the orientation. The pair (π, f) should satisfy the properness and fibrewise Morse conditions of Definition 3.9.*

There is a natural morphism ${}^\partial\mathcal{W} \to \mathcal{W}$ of sheaves, which is induced by the null-bordism of C. The reader is referred to [12] for a proof of the following result.

Proposition 8.3. — *The morphism ${}^\partial\mathcal{W} \to \mathcal{W}$ is a weak equivalence of sheaves.*

The definitions of all the sheaves of Section 7 generalize to the case where boundaries are permitted, as above. The analogue of Proposition 8.3 holds in each case: the null-bordism induces a natural morphism which is a weak equivalence. Henceforth, all such sheaves are taken with boundary, without addition of the decoration ∂.

The definition of $\mathcal{W}_T(X)$ becomes:

Definition 8.4. — *For T an object of $\mathcal{K}$, let $\mathcal{W}_T$ denote the sheaf with sections over X given by tuples $((V, \rho), q : M \to X, e)$ such that :*

1. *$(V, \rho) \in \mathcal{W}_{\mathrm{loc},T}$, of rank three;*
2. *$q : M \to X$ is a smooth bundle of closed 2-manifolds with boundary, with an orientation of the vertical tangent bundle;*
3. *there exists a neighbourhood of ∂M in M which identifies over X with a collar on $X \times C$, respecting the orientations;*

4. $e : D(V^\rho) \times_{T\times X} S(V^{-\rho}) \to M$ *is a smooth embedding over* X*, respecting orientations, with image disjoint from the boundary* ∂M*.*

8.2. Concordance lifting and fibres

Proposition 8.6 below identifies the homotopy fibre of the morphism $|\mathcal{W}_T| \to |\mathcal{W}_{\mathrm{loc},T}|$, by Corollary A.12, using the following concordance lifting property (see Definition A.9).

LEMMA 8.5. — *For* T *an object of* $\mathcal{K}$*, the forgetful morphism* $\mathcal{W}_T \to \mathcal{W}_{\mathrm{loc},T}$ *has the concordance lifting property.*

Fix an object (V, ρ) of $\mathcal{W}_{\mathrm{loc},T}(*)$; this is a three-dimensional oriented Morse vector bundle on the discrete space T. The canonical morphism $X \to *$ for a smooth manifold X, induces an associated object $(V, \rho)_X$ of $\mathcal{W}_{\mathrm{loc},T}(X)$, such that the structure is trivial with respect to X.

PROPOSITION 8.6. — *For* T *an object of* $\mathcal{K}$*, the fibre over* (V, ρ) *of the forgetful morphism* $\mathcal{W}_T \to \mathcal{W}_{\mathrm{loc},T}$ *is weakly equivalent to the sheaf which has sections over* X *the set of smooth bundles* $(q : M \to X)$ *of vertically tangentially oriented compact surfaces with collared boundary, where the boundary bundle* $\partial M \to X$ *identifies with*

$$-(C \amalg S(V^\rho) \times_T S(V^{-\rho})) \times X \to X.$$

Proof. — By cutting the interior of the embedded thickened spheres from the fibre bundle. □

8.3. The connectivity theorem

DEFINITION 8.7. — *For* T *an object of* $\mathcal{K}$*, let* $\mathcal{W}_{c,T}$ *denote the subsheaf of* $\mathcal{W}_T$ *which has sections over* X *the set of tuples* $((V, \rho), q : M \to X, e)$ *such that the fibres of* $M\backslash\mathrm{Im}(e) \to X$ *are connected.*

The construction is not functorial in T, since the extreme values $\{0, 3\}$ of the Morse index can lead to the introduction of non-connected fibres. This motivates the consideration of the following decomposition of the category $\mathcal{K}$.

DEFINITION 8.8. — *Let* $\mathcal{K}'$ *denote the full subcategory of* $\mathcal{K}$ *with objects such that the structure map* $T \to [\mathbf{3}]$ *has image contained in* $\{0, 3\}$ *and let* $\mathcal{K}''$ *be the full subcategory with objects such that the structure map* $T \to [\mathbf{3}]$ *has image contained in* $\{1, 2\}$*.*

The disjoint union of finite sets induces an equivalence of categories, $\mathcal{K}' \times \mathcal{K}'' \to \mathcal{K}$.

LEMMA 8.9. — *Let* Q *be an arbitrary object of* $\mathcal{K}$*, then the association* $S \mapsto \mathcal{W}_{c,Q\amalg S}$ *is contravariantly functorial in* $S \in \mathcal{K}''$*.*

The product decomposition of the category $\mathcal{K}$ implies the following result:

LEMMA 8.10. — *Let* $T \mapsto \mathcal{F}_T$ *be a functor from* $\mathcal{K}^{\mathrm{op}}$ *to sheaves on* $\mathfrak{X}$*, then there is a weak equivalence* $\mathrm{hocolim}_{T\in\mathcal{K}}|\mathcal{F}_T| \simeq \mathrm{hocolim}_{Q\in\mathcal{K}'}\mathrm{hocolim}_{S\in\mathcal{K}''}|\mathcal{F}_{Q\amalg S}|$.

The following technical result[12] avoids the necessity of considering the homotopy fibre of the homotopy colimit over $\mathcal{K}''$ of $\mathcal{W}_{c,Q\amalg S} \to \mathcal{W}_{Q\amalg S}$, for non-trivial Q. The reader is referred to [12] for the proof.

LEMMA 8.11. — *For any morphism $P \to Q$ in $\mathcal{K}'$, the following commutative square is homotopy cartesian:*

$$\begin{tikzcd}
\mathrm{hocolim}_{S\in\mathcal{K}''}|\mathcal{W}_{Q\amalg S}| \arrow[r] \arrow[d] & \mathrm{hocolim}_{S\in\mathcal{K}''}|\mathcal{W}_{P\amalg S}| \arrow[d] \\
\mathrm{hocolim}_{S\in\mathcal{K}''}|\mathcal{W}_{\mathrm{loc},Q\amalg S}| \arrow[r] & \mathrm{hocolim}_{S\in\mathcal{K}''}|\mathcal{W}_{\mathrm{loc},P\amalg S}|.
\end{tikzcd}$$

The connectivity theorem reads as follows:

THEOREM 8.12. — *The inclusion* $\mathrm{hocolim}_{S\in\mathcal{K}''}\ |\mathcal{W}_{c,S}| \to \mathrm{hocolim}_{S\in\mathcal{K}''}\ |\mathcal{W}_S|$ *is a weak equivalence.*

8.4. Indications on the proof of the connectivity theorem

The proof of Theorem 8.12 is by the construction of a homotopy inverse. Fibrewise surgery is used to make the fibres connected manifolds; this is achieved by cutting out pairs of disks and gluing in tubes. Such surgeries can be described in terms of the functoriality of the sheaves $\mathcal{W}_T$ with respect to $\mathcal{K}^{\mathrm{op}}$.

The proof uses methods of homotopical algebra to compare the surgery techniques which correspond to the choice of the regularization parameter $\{\pm 1\}$ for morphisms in the category $\mathcal{K}$. This reduces the proof to showing that the category which parametrizes the multiple surgeries which make a surface connected is contractible.

The reader is referred to [12] for the proof that the classifying space is contractible, which uses the criterion, derived from Proposition A.7, for a sheaf to be contractible.

9. THE PROOF OF THE MAIN THEOREM

Throughout this section, let C denote the manifold $S^1 \amalg -S^1$. The proof of Theorem 1.2 is completed by proving that the homotopy fibre of $|\mathcal{W}| \to |\mathcal{W}_{\mathrm{loc}}|$ is homotopy equivalent to $\mathbb{Z} \times B\Gamma^+_{\infty,2}$.

[12] A homotopy cartesian diagram corresponds to the homotopy limit, and is a derived version of the fibred product.

9.1. Reduction to integral homology

The Whitehead theorem implies that it is sufficient to construct a morphism

$$\mathbb{Z} \times B\Gamma_{\infty,2} \to \mathrm{hfib}\{|\mathcal{W}| \to |\mathcal{W}_{\mathrm{loc}}|\}$$

which induces an isomorphism in integral homology. Indeed, the morphism $|\mathcal{W}| \to |\mathcal{W}_{\mathrm{loc}}|$ is homotopic to a morphism of infinite loop spaces and therefore factorizes up to homotopy across Quillen's plus construction: $\mathbb{Z} \times B\Gamma^{+}_{\infty,2} \to \mathrm{hfib}\{|\mathcal{W}| \to |\mathcal{W}_{\mathrm{loc}}|\}$. This is a morphism between infinite loop spaces (not *a priori* an infinite loop map), which induces an isomorphism in integral homology; the Whitehead theorem implies that it is a weak equivalence.

9.2. Homotopy fibres for connected strata

The following result is fundamental; it follows from Proposition 8.6, restricted to the connected case.

Lemma 9.1. — *Let T be an object of $\mathcal{K}$, then the homotopy fibre over any basepoint of the localization map $|\mathcal{W}_{c,T}| \to |\mathcal{W}_{\mathrm{loc},T}|$ is homotopy equivalent to $\amalg_g B\Gamma_{g,2+2|T|}$.*

9.3. Stabilization via the genus

The collared boundary of the total space E of an object of $\mathcal{W}_T(X)$ allows for concatenation; this is a parametrized version of the concatenation used in the Harer-Ivanov cohomological stability theorems.

Definition 9.2. — *Let $z \in \mathcal{W}_{c,\emptyset}(*)$ be an oriented surface of genus one, with two boundary components. For T an object of $\mathcal{K}$:*

1. *let $.z : \mathcal{W}_T \to \mathcal{W}_T$ and $.z : \mathcal{W}_{c,T} \to \mathcal{W}_{c,T}$ denote the morphisms which are induced by concatenation with z;*
2. *let $z^{-1}\mathcal{W}_T$ (respectively $z^{-1}\mathcal{W}_{c,T}$) denote the colimit of the induced direct system of sheaves.*

The spaces represented by the colimits are weakly equivalent to the colimits of the individual spaces, by the following result:

Lemma 9.3. — [12] *There are natural weak equivalences $|z^{-1}\mathcal{W}_T| \simeq z^{-1}|\mathcal{W}_T|$, $|z^{-1}\mathcal{W}_{c,T}| \simeq z^{-1}|\mathcal{W}_{c,T}|$.*

Lemma 9.1 stabilizes to yield the following result:

Corollary 9.4. — *Let T be an object of $\mathcal{K}$, then the homotopy fibre of the localization map $|z^{-1}\mathcal{W}_{c,T}| \to |\mathcal{W}_{\mathrm{loc},T}|$ is homotopy equivalent to $\mathbb{Z} \times B\Gamma_{\infty,2+2|T|}$ over any basepoint.*

For Q a fixed object of $\mathcal{K}$, the associations $S \mapsto z^{-1}\mathcal{W}_S$, $S \mapsto z^{-1}\mathcal{W}_{c,Q\amalg S}$ are contravariantly functorial in S in $\mathcal{K}''$. Theorem 8.12 implies the following

Corollary 9.5. — *The map* $\mathrm{hocolim}_{S\in\mathcal{K}''}\ |z^{-1}\mathcal{W}_{c,S}| \to \mathrm{hocolim}_{S\in\mathcal{K}''}\ |z^{-1}\mathcal{W}_S|$ *is a weak equivalence.*

The realization spaces of $\mathcal{W}$ and $z^{-1}\mathcal{W}$ by the following Proposition, which follows from the fact that $|\mathcal{W}|$ is an infinite loop space (by Theorems 1.3 and 1.4).

Proposition 9.6. — *There are weak equivalences* $|\mathcal{W}| \simeq |z^{-1}\mathcal{W}| \simeq \mathrm{hocolim}_{T\in\mathcal{K}}|z^{-1}\mathcal{W}_T|$.

9.4. Morphisms and homology equivalences

For any small category $\mathcal{C}$ and any functor F_{-} from $\mathcal{C}$ to a suitable category of topological spaces, Spaces, there exists a canonical morphism $F_C \to \mathrm{hocolim}\ F_{-}$, for any object C of $\mathcal{C}$.

Lemma 9.7. — *Let Q be an object of $\mathcal{K}$. There is a canonical morphism*

$$\mathbb{Z}\times B\Gamma_{\infty,2+2|Q|} \to \mathrm{hfib}\{\mathrm{hocolim}_{S\in\mathcal{K}''}\ |z^{-1}\mathcal{W}_{c,Q\amalg S}| \to \mathrm{hocolim}_{S\in\mathcal{K}''}|\mathcal{W}_{\mathrm{loc},Q\amalg S}|\}.$$

Proof. — In the case $Q=\emptyset$, it suffices to observe that $\emptyset\in\mathcal{K}''$ and that $|z^{-1}\mathcal{W}_{c,\emptyset}|\simeq \mathbb{Z}\times B\Gamma_{\infty,2}$ and $|\mathcal{W}_{\mathrm{loc},\emptyset}|\simeq *$. The canonical morphism is the required morphism. The general case is a straightforward modification of this argument. □

The following technical result relates the homology of the homotopy fibre of a map between spaces obtained by gluing to the homology of the homotopy fibres of the maps between individual terms.

Proposition 9.8. — [12, 13] *Let $\mathcal{C}$ be a small category and let $u:\mathcal{G}_1\to\mathcal{G}_2$ be a natural transformation between functors $\mathcal{G}_i:\mathcal{C}\to$ Spaces.*

Suppose that, for any morphism $f:a\to b$ of $\mathcal{C}$, the map f_* from any homotopy fibre of u_a to the corresponding homotopy fibre of u_b induces an isomorphism in integral homology. Then, for any object a of $\mathcal{C}$, the inclusion of any homotopy fibre of u_a in the corresponding homotopy fibre of u_* : hocolim $\mathcal{G}_1 \to$ hocolim $\mathcal{G}_2$ induces an isomorphism in integer homology.

Proposition 9.9. — *The canonical morphism*

$$\mathbb{Z}\times B\Gamma_{\infty,2} \to \mathrm{hfib}\{\mathrm{hocolim}_{S\in\mathcal{K}''}\ |z^{-1}\mathcal{W}_{c,S}| \to \mathrm{hocolim}_{S\in\mathcal{K}''}|\mathcal{W}_{\mathrm{loc},S}|\}$$

induces an isomorphism in integral homology.

Proof. — There is a commutative diagram of homotopy fibre sequences, for $S\to T$ a morphism of $\mathcal{K}''$:

$$\begin{array}{ccccc}
\mathbb{Z}\times B\Gamma_{\infty,2+2|T|} & \longrightarrow & |z^{-1}\mathcal{W}_{c,T}| & \longrightarrow & |\mathcal{W}_{\mathrm{loc},T}| \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}\times B\Gamma_{\infty,2+2|S|} & \longrightarrow & |z^{-1}\mathcal{W}_{c,S}| & \longrightarrow & |\mathcal{W}_{\mathrm{loc},S}|
\end{array}$$

where the map of homotopy fibres corresponds geometrically to attaching tubes $D^1\times S^1$ or pairs of disks $D^2\times S^0$, according to the regularization index. The Harer-Ivanov stability theorems imply that these morphisms induce an isomorphism in homology. The result follows by applying Proposition 9.8. □

9.5. Applying the homotopy cartesian square

Lemma 8.11 stabilizes to give the following result.

LEMMA 9.10. — *For any morphism $P \to Q$ in $\mathcal{K}'$, the following commutative square is homotopy cartesian:*

$$\begin{array}{ccc}
\mathrm{hocolim}_{S\in\mathcal{K}''}|z^{-1}\mathcal{W}_{Q\amalg S}| & \longrightarrow & \mathrm{hocolim}_{S\in\mathcal{K}''}|z^{-1}\mathcal{W}_{P\amalg S}| \\
\downarrow & & \downarrow \\
\mathrm{hocolim}_{S\in\mathcal{K}''}|\mathcal{W}_{\mathrm{loc},Q\amalg S}| & \longrightarrow & \mathrm{hocolim}_{S\in\mathcal{K}''}|\mathcal{W}_{\mathrm{loc},P\amalg S}|
\end{array}$$

This allows the deduction of the following result.

LEMMA 9.11. — *For any object Q of $\mathcal{K}$, the canonical morphism*

$$\mathbb{Z}\times B\Gamma_{\infty,2+2|Q|} \to \mathrm{hfib}\{\mathrm{hocolim}_{S\in\mathcal{K}''}\ |z^{-1}\mathcal{W}_{Q\amalg S}| \to \mathrm{hocolim}_{S\in\mathcal{K}''}|\mathcal{W}_{\mathrm{loc},Q\amalg S}|\}$$

induces an isomorphism in integral homology.

Proof. — There is a commutative diagram

$$\begin{array}{ccccccc}
\mathbb{Z}\times B\Gamma_{\infty,2+2|Q|} & \longrightarrow & F & \longrightarrow & \mathrm{hocolim}_{S\in\mathcal{K}''}|z^{-1}\mathcal{W}_{Q\amalg S}| & \longrightarrow & \mathrm{hocolim}_{S\in\mathcal{K}''}|\mathcal{W}_{\mathrm{loc},Q\amalg S}| \\
\downarrow & & \simeq\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}\times B\Gamma_{\infty,2} & \longrightarrow & F' & \longrightarrow & \mathrm{hocolim}_{S\in\mathcal{K}''}|z^{-1}\mathcal{W}_{S}| & \longrightarrow & \mathrm{hocolim}_{S\in\mathcal{K}''}|\mathcal{W}_{\mathrm{loc},S}|
\end{array}$$

in which F, F' denote the respective homotopy fibres, which are weakly equivalent, since the right hand square is homotopy cartesian.

The left hand vertical morphism is an integral homology isomorphism, by the Harer-Ivanov stability theorems and the bottom left hand morphism induces an isomorphism in integral homology, by the result of the previous Section and Corollary 9.4, which implies that the term $\mathrm{hocolim}_{S\in\mathcal{K}''}\ |z^{-1}\mathcal{W}_S|$ is equivalent to $\mathrm{hocolim}_{S\in\mathcal{K}''}\ |z^{-1}\mathcal{W}_{c,S}|$.

The result follows. □

9.6. Proof of the main theorem

The proof of the Theorem 1.5 is completed by repeating the above argument, using Proposition 9.8 together with the Harer-Ivanov stability theorem, applied to the homotopy fibre of the morphism:

$$\mathrm{hocolim}_{Q\in\mathcal{K}'}\mathrm{hocolim}_{S\in\mathcal{K}''}\ |z^{-1}\mathcal{W}_{Q\amalg S}| \to \mathrm{hocolim}_{Q\in\mathcal{K}'}\mathrm{hocolim}_{S\in\mathcal{K}''}|\mathcal{W}_{\mathrm{loc},Q\amalg S}|.$$

APPENDIX A
HOMOTOPY THEORY OF SHEAVES

Throughout this section, all sheaves are defined on the category $\mathfrak{X}$ of smooth manifolds.

A.1. Concordance for sheaves

Definition A.1. — *Let $\mathcal{F}$ be a sheaf and let X be a smooth manifold. Two sections $s_0, s_1 \in \mathcal{F}(X)$ are concordant if there exists a section $s \in \mathcal{F}(X \times \mathbb{R})$ such that $s = p_X^* s_0$ on an open neighbourhood of $X \times (-\infty, 0]$ and $s = p_X^* s_1$ on an open neighbourhood of $X \times [1, \infty)$, where $p_X : X \times \mathbb{R} \to X$ is the projection. The concordance s is said to start at s_0.*

Concordance defines an equivalence relation on the set $\mathcal{F}(X)$; the set of equivalence classes modulo concordance is written $\mathcal{F}[X]$.

There is a relative version of concordance with respect to a closed subset $A \subset X$ (not necessarily a manifold) of a smooth manifold X.

Definition A.2. — *For $A \subset X$ a closed subset of a smooth manifold X in $\mathfrak{X}$,*

1. *let $\mathcal{F}_A$ denote the colimit $\mathcal{F}_A := \mathrm{colim}_U \mathcal{F}(U)$, where U ranges over the category of open neighbourhoods of A in X;*
2. *for $s \in \mathcal{F}_A$, let $\mathcal{F}(X, A; s)$ denote the fibre of the canonical morphism $\mathcal{F}(X) \to \mathcal{F}_A$ above s.*

Definition A.3. — *For $\mathcal{F}, X, A, s$ as above, elements $t_0, t_1 \in \mathcal{F}(X, A; s)$ are concordant relative to A if there exists an element $t \in \mathcal{F}(X \times \mathbb{R}, A \times \mathbb{R}; q^* s)$ which defines a concordance for t_0, t_1 regarded as elements of $\mathcal{F}(X)$, where $q^* s \in \mathcal{F}_{A \times \mathbb{R}}$ is induced by pullback of s via the projection $q : A \times \mathbb{R} \to A$.*

Let $\mathcal{F}[X, A; s]$ denote the set $\mathcal{F}(X, A; s)$ modulo the equivalence relation given by concordance relative to A.

A.2. The representing space $|\mathcal{F}|$

Let $\boldsymbol{\Delta}^n \subset \mathbb{R}^{n+1}$ denote the affine plane $\Sigma x_i = 1$, so that $n \mapsto \boldsymbol{\Delta}^n$ defines a cosimplicial smooth manifold.

Definition A.4. — *For $\mathcal{F}$ a sheaf of sets, let $|\mathcal{F}|$ denote the geometric realization of the simplicial set $\mathcal{F}(\boldsymbol{\Delta}^*)$.*

A point $z \in \mathcal{F}(*)$ induces a point in $|\mathcal{F}|$. The following representability statement is fundamental.

Proposition A.5. — [12] *For any point $z \in \mathcal{F}(*)$, smooth manifold X and closed subset $A \subset X$, there is a natural isomorphism $\mathcal{F}[X, A; z] \cong [(X, A), (|\mathcal{F}|, z)]$, where the right hand side denotes homotopy classes of maps of pairs.*

The following formal properties are basic.

1. The functor $\mathcal{F} \mapsto |\mathcal{F}|$ takes pullback squares of sheaves to pullbacks of compactly-generated Hausdorff spaces.
2. $|\mathcal{F}_1 \amalg \mathcal{F}_2| \cong |\mathcal{F}_1| \amalg |\mathcal{F}_2|$.

Definition A.6. — *A morphism of sheaves of sets $v : \mathcal{E} \to \mathcal{F}$ is a weak equivalence if $|v| : |\mathcal{E}| \to |\mathcal{F}|$ is a weak equivalence of topological spaces. This is equivalent to the induced morphism $\pi_n(|\mathcal{E}|, z) \cong \mathcal{E}[S^n, e; z] \to \pi_n(|\mathcal{F}|, z) \cong \mathcal{F}[S^n, e; z]$ being an isomorphism, for all n, z.*

A.3. Special weak equivalences

The following special weak equivalences resemble trivial fibrations in homotopical algebra.

Proposition A.7. — [12] *Let $v : \mathcal{E} \to \mathcal{F}$ be a morphism of sheaves of sets. Suppose that v induces a surjection $\mathcal{E}[X, A; s] \to \mathcal{F}[X, A; v(s)]$ for all smooth X and all $s \in \mathcal{E}_A$, then v is a weak equivalence.*

Example A.8. — Let $\mathcal{F}$ be the terminal sheaf which is induced by the constant presheaf with value a singleton set; for a sheaf of sets, $\mathcal{E}$, there is a canonical morphism of sheaves, $\mathcal{E} \to \mathcal{F}$. The morphism induces a surjection on relative concordance classes, as in the Proposition above, if and only if any section $s \in \mathcal{E}_A$ extends to an element of $\mathcal{E}[X, A; s]$. This yields a criterion for a sheaf to be homotopically trivial.

A.4. The concordance lifting property

The concordance lifting property plays the rôle of the fibration hypothesis in homotopical algebra; it is important in considering the homotopy fibre of a morphism of sheaves (see Corollary A.12 below).

Definition A.9. — *A morphism $u : \mathcal{E} \to \mathcal{F}$ of sheaves has the concordance lifting property if, for any section $s \in \mathcal{E}(X)$ and concordance $h \in \mathcal{F}(X \times \mathbb{R})$ starting at $u(s)$, there exists a concordance $H \in \mathcal{E}(X \times \mathbb{R})$, starting at s, which lifts h.*

Proposition A.10. — [12] *Let $\mathcal{E} \times_{\mathcal{G}} \mathcal{F}$ be the pullback of the diagram of sheaves, $\mathcal{E} \to \mathcal{G} \xleftarrow{u} \mathcal{F}$. If u has the concordance lifting property, then so does $\mathcal{E} \times_{\mathcal{G}} \mathcal{F} \to \mathcal{E}$ and the associated diagram of spaces,*

$$\begin{tikzcd} |\mathcal{E} \times_{\mathcal{G}} \mathcal{F}| \arrow[r] \arrow[d] & |\mathcal{F}| \arrow[d, "|u|"] \\ |\mathcal{E}| \arrow[r] & |\mathcal{G}|, \end{tikzcd}$$

is homotopy cartesian.

Definition A.11. — *For $u : \mathcal{E} \to \mathcal{F}$ a morphism of sheaves and $a \in \mathcal{F}(*)$, let $\mathcal{E}_a$ denote the fibre of u over a, which is the sheaf defined by $\mathcal{E}_a(X) := \{s \in \mathcal{E}(X) | u(s) = a\}$.*

Corollary A.12. — [12] *Let $u : \mathcal{E} \to \mathcal{F}$ be a morphism of sheaves which has the concordance lifting property, then the induced diagram of spaces $|\mathcal{E}_a| \to |\mathcal{E}| \to |\mathcal{F}|$ is a homotopy fibre sequence.*

A.5. Functors to small categories

Let $\mathcal{F}$ be a sheaf with values in the category of small categories. The nerve construction [20] $N_\bullet\mathcal{F}$ defines a sheaf with values in the category of simplicial sets; hence $N_\bullet\mathcal{F}(\mathbf{\Delta}^*)$ has the structure of a bisimplicial set. This can be realized as a topological space in a number of ways; in particular, the topological realization of the diagonal simplicial set is weakly equivalent to the classifying space $B|\mathcal{F}|$ of the topological category $|\mathcal{F}|$. (Here $|\mathcal{F}|$ is the topological category with object space $|N_0\mathcal{F}|$ and morphism space $|N_1\mathcal{F}|$).

Definition A.13. — [12] *For $\mathcal{F}$ as above, let $\beta\mathcal{F}$ denote the sheaf of sets which has sections over X the set of pairs $(\mathfrak{Y}, \phi_{\bullet\bullet})$, where $\mathfrak{Y} := \{Y_j | j \in \mathcal{J}\}$ is a locally finite open cover of X, indexed by a fixed index set $\mathcal{J}$, and, for nonempty finite subsets $R \subset S$, $\phi_{RS} \in N_1\mathcal{F}(Y_S)$ is a morphism subject to cocycle conditions, where Y_S denotes the open set $Y_S := \bigcap_{j\in S} Y_j$.*

The following theorem is proved in [12] and gives a sheaf theoretic model for the classifying space $B|\mathcal{F}|$.

Theorem A.14. — *The spaces $|\beta\mathcal{F}|$ and $B|\mathcal{F}|$ are homotopy equivalent.*

A.6. Homotopy colimits

Homotopy colimits are used for gluing; they behave as the derived functor of the colimit and, in particular, satisfy a homotopy invariance property.

Definition A.15. — *Let $\mathcal{F}_C$ be a sheaf of sets, functorial in objects C of a small, discrete category $\mathcal{C}$.*

1. *Let $\mathcal{C} \int \mathcal{F}$ denote the sheaf with values in small categories which associates to X the small category with objects pairs (C, w), where C is an object of $\mathcal{C}$ and $w \in \mathcal{F}_C(X)$. The morphisms are defined in the usual way.*
2. *Let* $\mathrm{hocolim}_{C\in\mathcal{C}}|\mathcal{F}_C|$ *be the space*
$$B|\mathcal{C}\int \mathcal{F}_\bullet| \simeq |\beta(\mathcal{C}\int \mathcal{F}_\bullet)|.$$

For $\mathcal{F}$, $\mathcal{C}$ as above, it makes sense to define the homotopy colimit as a sheaf:
$$\mathrm{hocolim}_{C\in\mathcal{C}}\mathcal{F}_C := \beta(\mathcal{C}\int \mathcal{F}_\bullet).$$
The structure of this object is made explicit in [12].

Definition A.16. — *Let $g : \mathcal{E} \to \mathcal{F}$ be a natural transformation of sheaves of small categories. The natural transformation, g, is a transport projection if the square*
$$\begin{tikzcd} N_1\mathcal{E} \arrow[r] \arrow[d] & N_0\mathcal{E} \arrow[d] \\ N_1\mathcal{F} \arrow[r] & N_0\mathcal{F} \end{tikzcd}$$

is a cartesian square of sheaves of sets.

The following gluing result for homotopy colimits allows the passage from local to global.

Proposition A.17. — [12] *Let $g : \mathcal{E} \to \mathcal{F}$, $g' : \mathcal{E}' \to \mathcal{F}$ be transport projections between sheaves of small categories and let $u : \mathcal{E} \to \mathcal{E}'$ be a morphism of sheaves of small categories above $\mathcal{F}$. Suppose that*

1. *the morphisms $N_0\mathcal{E} \to N_0\mathcal{F}$, $N_0\mathcal{E}' \to N_0\mathcal{F}$ have the concordance lifting property;*
2. *for all $a \in \mathcal{F}(*)$, the morphism $N_0\mathcal{E}_a \to N_0\mathcal{E}'_a$ induces a weak equivalence (respectively an integral homology equivalence).*

Then the morphism $\beta u : \beta\mathcal{E} \to \beta\mathcal{E}'$ is a weak equivalence (respectively induces an integral homology equivalence).

REFERENCES

[1] T. Bröcker & K. Jänich – *Introduction to differential topology*, Cambridge University Press, Cambridge, 1982, Translated from the German by C. B. Thomas and M. J. Thomas.

[2] C. J. Earle & J. Eells – "A fibre bundle description of Teichmüller theory", *J. Differential Geometry* **3** (1969), p. 19–43.

[3] C. J. Earle & A. Schatz – "Teichmüller theory for surfaces with boundary", *J. Differential Geometry* **4** (1970), p. 169–185.

[4] S. Galatius – "Mod p homology of the stable mapping class group", *Topology* **43** (2004), no. 5, p. 1105–1132.

[5] R. Hain & E. Looijenga – "Mapping class groups and moduli spaces of curves", in *Algebraic geometry—Santa Cruz 1995* (Providence, RI), Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., 1997, p. 97–142.

[6] J. Harer – "The second homology group of the mapping class group of an orientable surface", *Invent. Math.* **72** (1983), no. 2, p. 221–239.

[7] ———, "Stability of the homology of the mapping class groups of orientable surfaces", *Ann. of Math. (2)* **121** (1985), no. 2, p. 215–249.

[8] J. Harris & I. Morrison – *Moduli of curves*, Graduate Texts in Mathematics, vol. 187, Springer-Verlag, New York, 1998.

[9] N. V. Ivanov – "Stabilization of the homology of Teichmüller modular groups", *Algebra i Analiz* **1** (1989), no. 3, p. 110–126.

[10] I. Madsen & R. J. Milgram – *The classifying spaces for surgery and cobordism of manifolds*, Annals of Mathematics Studies, vol. 92, Princeton University Press, Princeton, N.J., 1979.

[11] I. Madsen & U. Tillmann – "The stable mapping class group and $Q(\mathbb{C}P^\infty_+)$", *Invent. Math.* **145** (2001), no. 3, p. 509–544.

[12] I. Madsen & M. Weiss – "The stable moduli space of Riemann surfaces: Mumford's conjecture", *Preprint* (2004), p. 97.

[13] D. McDuff & G. Segal – "Homology fibrations and the "group-completion" theorem", *Invent. Math.* **31** (1975/76), no. 3, p. 279–284.

[14] E. Y. Miller – "The homology of the mapping class group", *J. Differential Geom.* **24** (1986), no. 1, p. 1–14.

[15] S. Morita – "Characteristic classes of surface bundles", *Invent. Math.* **90** (1987), no. 3, p. 551–577.

[16] ______, "Structure of the mapping class groups of surfaces: a survey and a prospect", in *Proceedings of the Kirbyfest (Berkeley, CA, 1998)*, Geom. Topol. Monogr., vol. 2, Geom. Topol. Publ., Coventry, 1999, p. 349–406 (electronic).

[17] D. Mumford – "Towards an enumerative geometry of the moduli space of curves", in *Arithmetic and geometry, Vol. II*, Progr. Math., vol. 36, Birkhäuser Boston, Boston, MA, 1983, p. 271–328.

[18] A. Phillips – "Submersions of open manifolds", *Topology* **6** (1967), p. 171–206.

[19] D. Quillen – "Elementary proofs of some results of cobordism theory using Steenrod operations", *Advances in Math.* **7** (1971), p. 29–56 (1971).

[20] G. Segal – "Categories and cohomology theories", *Topology* **13** (1974), p. 293–312.

[21] R. E. Stong – *Notes on cobordism theory*, Mathematical notes, Princeton University Press, Princeton, N.J., 1968.

[22] U. Tillmann – "On the homotopy of the stable mapping class group", *Invent. Math.* **130** (1997), no. 2, p. 257–275.

[23] V. A. Vassiliev – "Topology of spaces of functions without complicated singularities", *Funktsional Anal. i Prilozhen* **93** (1989), p. 24–36.

[24] ______, *Complements of discriminants of smooth maps: topology and applications*, Translations of Mathematical Monographs, vol. 98, American Mathematical Society, Providence, RI, 1992, Translated from the Russian by B. Goldfarb.

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