Math 464/564 : Algebra III
Problem Set 11

1. This problem is intended to give you practice in chasing diagrams. Consider a commutative diagram with exact rows

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "h_1"'] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"'] & A_3 \arrow[r, "f_3"] \arrow[d, "h_3"'] & A_4 \arrow[r, "f_4"] \arrow[d, "h_4"'] & A_5 \arrow[d, "h_5"'] \\ B_1 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & B_3 \arrow[r, "g_3"] & B_4 \arrow[r, "g_4"] & B_5 \end{tikzcd}$$

   Assume that h_1 and h_2 and h_4 and h_5 are isomorphisms. Prove that h_3 is an isomorphism (this is usually called the *five lemma*).

2. (a) Regard $\mathbb{F}$ as a $\mathbb{F}[x]$ module where x acts as 0. Construct a free resolution of $\mathbb{F}$ over $\mathbb{F}[x]$.
   (b) Calculate the following, where $P_n = \mathbb{F}[x]/(x^n)$.
      i. $\mathrm{Tor}_*^{\mathbb{F}[x]}(\mathbb{F},\mathbb{F})$ and $\mathrm{Ext}^*_{\mathbb{F}[x]}(\mathbb{F},\mathbb{F})$.
      ii. $\mathrm{Tor}_*^{\mathbb{F}[x]}(\mathbb{F},P_n)$ and $\mathrm{Ext}^*_{\mathbb{F}[x]}(\mathbb{F},P_n)$.

3. For any $f \in \mathbb{F}[x_1,\ldots,x_n]$, prove that $\{f\}$ is a Grobner basis for the principal ideal (f).

4. Let $I = (x^{\alpha(0)},\ldots,x^{\alpha(k)})$ be a monomial ideal in $\mathbb{F}[x_1,\ldots,x_n]$ for some multi-indices $\alpha(0),\ldots,\alpha(k)$. Consider a polynomial $f \in I$, and let $c_\beta x^\beta$ be a term in f with $c_\beta \neq 0$. Prove that x^β is divisible by one of the $\alpha(i)$.

5. Fix real numbers $w_1,\ldots,w_n$. For a monomial $x^\alpha \in \mathbb{F}[x_1,\ldots,x_n]$ with $\alpha = (\alpha_1,\ldots,\alpha_n)$, define $w(x^\alpha) = w_1\alpha_1 + \cdots + w_n\alpha_n$. Order the monomials in $\mathbb{F}[x_1,\ldots,x_n]$ such that $x^\alpha > x^\beta$ if and only if $w(x^\alpha) > w(x^\beta)$. This is called a *weight ordering*.
   (a) Take $n = 2$ and $w_1 = 3$ and $w_2 = 7$. Is the associated weight ordering a monomial ordering?
   (b) Take $n = 2$ and $w_1 = 1$ and $w_2 = \pi$. Is the associated weight ordering a monomial ordering?
   (c) Give necessary and sufficient conditions on the weights w_i for a weight ordering to be a monomial ordering.
   (d) Can lexicographic order be given as a weight ordering?

6. Prove that there is a unique monomial ordering on $\mathbb{C}[x]$.

7. (a) Give an example of a monomial ideal in $\mathbb{C}[x,y]$ with a minimal set of generators consisting of five elements.
   (b) Is there any bound on the number of generators of a monomial ideal in $\mathbb{C}[x,y]$?

8. Show that $\{x_1 - x_2^{37}, x_1 - x_2^{38}\}$ is not a Grobner basis with respect to lexicographic order.