Math 464/564 : Algebra III
Problem Set 2

Remark. In this problem set, if $V \subset \mathbb{F}^n$ is a variety then we will call the ring $\mathbb{F}[z_1, \ldots, z_n]/I(V)$ of polynomial functions on V the *affine coordinate ring* of V.

1. Math 564 students need to do Atiyah-Macdonald Chapter 1, problems 11 and 14. Everyone has to do the following problems.

2. In this problem, we will consider varieties in $\mathbb{F}^2$ defined by polynomials in $\mathbb{F}[x, y]$.

   (a) Prove that the affine coordinate ring of the variety $V(y-x^2)$ is isomorphic to a polynomial ring in one variable over $\mathbb{F}$.

   (b) Prove that the affine coordinate ring of the variety $V(xy - 1)$ is not isomorphic to a polynomial ring in one variable over $\mathbb{F}$.

3. Define
   $$Y = \{(t, t^2, t^3) \mid t \in \mathbb{F}\} \subset \mathbb{F}^3.$$
   Prove that Y is a variety, and find generators for $I(Y)$. Finally, prove that the affine coordinate ring of Y is isomorphic to a polynomial ring in one variable over $\mathbb{F}$.

4. Let $Y = V(z_1^2 - z_2 z_3, z_1 z_3 - z_1) \subset \mathbb{F}^3$. Draw a picture of Y in the special case $\mathbb{F} = \mathbb{R}$. Prove that Y is a union of three irreducible components, and find generators for the prime ideals defining these components.

5. Prove that a $\mathbb{F}$-algebra A is the affine coordinate ring of an algebraic variety if and only if it is finitely generated and has no nilpotent elements (if you don't know the definition of a $\mathbb{F}$-algebra, I recommend the wikipedia article "Algebra over a field").

6. In this exercise, you'll be guided through a quick and dirty proof of the Nullstellensatz for the field $\mathbb{C}$. The solution should consist of proofs of the "problems" that are interspersed. In fact, what we'll prove is that every maximal ideal of $\mathbb{C}[z_1, \ldots, z_n]$ is of the form $(z_1 - a_1, \ldots, z_n - a_n)$ for some $a_1, \ldots, a_n \in \mathbb{C}$. Consider any maximal ideal M of $\mathbb{C}[z_1, \ldots, z_n]$. It is enough to prove that for all $1 \leq i \leq n$, there exists some $a_i \in \mathbb{C}$ such that $z_i - a_i \in M$, since then $(z_1 - a_1, \ldots, z_n - a_n) \subset M$. The fact that $(z_1 - a_1, \ldots, z_n - a_n)$ is maximal will then imply that we have equality.

   Let K be the field $\mathbb{C}[z_1, \ldots, z_n]/M$. We have a projection $\mathbb{C}[z_1, \ldots, z_n] \to K$. Let $\pi_i : \mathbb{C}[z_i] \to K$ be the restriction of this projection to $\mathbb{C}[z_i] \subset \mathbb{C}[z_1, \ldots, z_n]$. Our goal is to find some $a_i \in \mathbb{C}$ such that $z_i - a_i \in \ker(\pi_i)$.

   Problem. *Observe that K is a vector space over $\mathbb{C}$. Prove that K is at most countable dimensional as a vector space over $\mathbb{C}$.*

   Problem. *Let $\mathbb{C}(z)$ be the field of rational functions in one variable over $\mathbb{C}$, i.e. the field consisting of functions of the form $\frac{f(z)}{g(z)}$, where $f(z), g(z) \in \mathbb{C}[z]$ and $g(z) \neq 0$. The field $\mathbb{C}(z)$ is a vector space over $\mathbb{C}$. Prove that its dimension is uncountable. Hint : show that the set $\{\frac{1}{z-a} \mid a \in \mathbb{C}\}$ is linearly independent.*

   Problem. *Use the previous two problems to show that $\ker(\pi_i) \neq 0$. Hint : If it is 0, then we can find a copy of $\mathbb{C}[z_i]$ in K. Why does this mean we can find a copy of $\mathbb{C}(z_i)$ in K?*

   Problem. *Prove that $\ker(\pi_i)$ contains a linear term of the form $z_i - a_i$ for some $a_i \in \mathbb{C}$.*