Math 366 : Geometry
Problem Set 2

1. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is *linear* if $f(\vec{v} + \vec{w}) = f(\vec{v}) + f(\vec{w})$ for all $\vec{v}, \vec{w} \in \mathbb{R}^n$ and if $f(c\vec{v}) = cf(\vec{v})$ for all $\vec{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. A function $g : \mathbb{R}^n \to \mathbb{R}^n$ is *affine* if there exists a linear function $f : \mathbb{R}^n \to \mathbb{R}^n$ and a point $\vec{v}_0 \in \mathbb{R}^n$ such that $g(\vec{w}) = f(\vec{w}) + \vec{v}_0$ for all $\vec{w} \in \mathbb{R}^n$.

   (a) If $g : \mathbb{R}^n \to \mathbb{R}^n$ is affine and $U \subset \mathbb{R}^n$ is convex, then prove that $g(U)$ is convex.

   (b) If $g : \mathbb{R}^n \to \mathbb{R}^n$ is affine and $V \subset \mathbb{R}^n$ is convex, then prove that $g^{-1}(V)$ is convex.

2. Assume that $x_1, \ldots, x_n \in \mathbb{R}^2$ are distinct point such that for any $1 \leq i < j < k < \ell \leq n$, the four points $\{x_i, x_j, x_k, x_\ell\}$ form the vertices of a convex 4-gon. Prove that the points $\{x_1, \ldots, x_n\}$ form the vertices of a convex n-gon.

3. Give a rigorous proof that if S is a set of 5 distinct points in general position in $\mathbb{R}^2$, then four points in S form the vertices of a convex 4-gon (i.e. a quadrilateral). Make sure that your proof is complete and rigorous; don't just draw some pictures and assert that they exhaust all possibilities.

4. Does Helly's theorem hold for infinite numbers of convex sets? In other words, if $U_1, U_2, \ldots$ are convex sets in $\mathbb{R}^d$ such that any k-fold intersection of the U_i's is nonempty for $k \leq d+1$, then is it necessarily true that $\cap_{i=1}^{\infty} U_i$ is nonempty? Either prove this or construct a counterexample. Hint : Think hard about what happens in $\mathbb{R}^1$.

5. Consider $n \geq 4$ parallel line segments in $\mathbb{R}^2$. Assume that for every three of these line segments, there is a line in $\mathbb{R}^2$ meeting all three segments. Prove that there is a single line meeting all n of the line segments. Hint : First rotate the plane so that all the line segments in question are vertical. For any vertical line segment S, let

$$C(S) = \{(\alpha, \beta) \in \mathbb{R}^2 \mid \text{the line } y = \alpha x + \beta \text{ meets } S\}.$$

   First prove that $C(S)$ is convex, and then use this together with Helly's theorem to prove the desired result.