Math 366 : Geometry
Problem Set 3

1. Say that a finite set X of points in $\mathbb{R}^2$ is *extra-special* if all but one point of X lies on a line. Prove that if $X \subset \mathbb{R}^2$ is a finite set of points not all of which lie on a line such the set
$$\{\ell \mid \ell \text{ is a line passing through 2 points of } X\}$$
has size $|X|$, then X is extra-special. Hint : Carefully examine the proof of the Erdos-de Bruijn theorem.

2. Define
$$\mathcal{P} = \{(a,b) \in \mathbb{R}^2 \mid (a,b) \neq (0,0)\},$$
$$\mathcal{L} = \{\ell \mid \ell \text{ a line in } \mathbb{R}^2 \text{ not passing through } (0,0)\}.$$

   (a) Prove that every element of $\mathcal{L}$ can be uniquely expressed in the form $ax + by = 1$ for some $a, b \in \mathbb{R}$ with $(a,b) \neq (0,0)$.

   (b) Define functions
$$\psi : \mathcal{P} \longrightarrow \mathcal{L}$$
$$\psi(a,b) = \text{the line } ax + by = 1$$
   and
$$\phi : \mathcal{L} \longrightarrow \mathcal{P}$$
$$\phi(\ell) = (a,b), \text{ where } \ell \text{ is the line } ax + by = 1.$$
   Prove that ψ and ϕ are well-defined bijections.

   (c) Prove that for $(a,b) \in \mathcal{P}$ and $\ell \in \mathcal{L}$, the point (a,b) lies on ℓ if and only if the point $\phi(\ell)$ lies on the line $\psi(a,b)$.

   (d) Prove that if the line ℓ determined by two distinct points $(a_1,b_1),(a_2,b_2) \in \mathcal{P}$ does not pass through $(0,0)$ (i.e. if $\ell \in \mathcal{L}$), then the lines $\psi(a_1,b_1)$ and $\psi(a_2,b_2)$ intersect in the point $\phi(\ell)$.

   (e) Prove that if the lines $\ell_1, \ell_2 \in \mathcal{L}$ intersect in a point $(a,b) \in \mathcal{P}$, then the line $\psi(a,b)$ passes through the points $\phi(\ell_1)$ and $\phi(\ell_2)$.

3. Prove the *dual* Sylvester-Gallai theorem : If X is a finite set of lines in $\mathbb{R}^2$ that do not all intersect in a single point and none of the lines in X are parallel, then there exists some point in $\mathbb{R}^2$ lying in exactly two of the lines in X. Hint : First translate the lines in X so that none of them pass through $(0,0)$. Letting the notation be as in the previous problem, argue that we can apply the ordinary Sylvester-Gallai theorem to $\phi(X)$ (make sure it satisfies the conditions of the Sylvester-Gallai theorem!). We thus obtain some line ℓ passing through exactly two points of $\phi(X)$. Prove that ℓ does not pass through $(0,0)$ (this should use the fact that none of the lines in X are parallel!), so we can apply ϕ to it. Finally, prove that $\phi(\ell)$ is the point we are looking for.