Math 366 : Geometry
Problem Set 4

For problems 2-5, you will get half credit for a proof in $\mathbb{R}^2$.

1. Let $D \subset \mathbb{R}^2$ be a disc of radius $\frac{1}{\sqrt{3}}$ centered at the origin. For $\epsilon > 0$, define

$$X_\epsilon = \{(x, y) \in D \mid -\frac{1}{\sqrt{3}} \leq x \leq \frac{1}{\sqrt{3}} - \epsilon\}.$$

   Prove that for $0 < \epsilon < 0.1$, the set X_ϵ can be subdivided into three regions each of diameter strictly less than 1.

2. Let ℓ be a finite line segment in $\mathbb{R}^n$ and let $p \in \mathbb{R}^n$ be a point. Prove that the point $q \in \ell$ whose distance from p is maximal is one of the endpoints of ℓ.

3. Consider $\{x_1, \ldots, x_k\} \subset \mathbb{R}^n$ which form the vertices of a convex polytope (this means that for all $1 \leq i \leq k$, the point x_i is not in the convex hull of the other points $\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k\}$). For a point $p \in \text{conv}(x_1, \ldots, x_k)$, define the *weight* of p to be the number of nonzero t_i in an expression $p = t_1 x_1 + \cdots + t_k x_k$ with $t_i \geq 0$ and $t_1 + \cdots + t_k = 1$.

   Problem : Prove that if the weight of $p \in \text{conv}(x_1, \ldots, x_k)$ is strictly greater than 1, then there exists a line segment ℓ in $\text{conv}(x_1, \ldots, x_k)$ containing p such that the endpoints of ℓ have weight strictly smaller than p. Hint : Writing $p = t_1 x_1 + \cdots + t_k x_k$, choose x_i such that $t_i \neq 0$. The line segment you want will start at x_i and go through p (and, of course, end at a point of smaller weight).

4. Consider $\{x_1, \ldots, x_k\} \subset \mathbb{R}^n$ which form the vertices of a convex polytope, and let d be the diameter of the convex hull of the x_i. Prove that

$$d = \max\{\text{dist}(x_i, x_j) \mid 1 \leq i, j \leq k\}.$$

   Hint : Consider two points p_1 and p_2 in the convex hull of the x_i. Prove that if one of the p_i (say, p_1) is not equal to one of the x_i, then there exists a point p_1' in the convex hull whose weight is strictly smaller than the weight of p_1 such that $\text{dist}(p_1, p_2) < \text{dist}(p_1', p_2)$. You will use problems 2 and 3. Why does this imply the desired result?

5. Prove that if $X \subset \mathbb{R}^n$ is a box of the form $[x_1, y_1] \times [x_2, y_2] \times \cdots \times [x_n, y_n]$ with $x_i < y_i$ for all i and the diameter of X is 1, then X can be subdivided into two parts whose diameters are strictly smaller than 1. Half credit for a 2-dimensional proof.