Math 366 : Geometry
Problem Set 5

1. Let $P \subset \mathbb{R}^2$ be a convex polygon of diameter 1. Prove that there is a circle of radius $1/\sqrt{3}$ that divides P into two equal-area regions. Hint : You'll want to use the intermediate value theorem somehow. You'll also need to apply a theorem that we proved a while ago (where does the number $1/\sqrt{3}$ appear?).

2. Give (with proof!) an example of a convex polygon P in $\mathbb{R}^2$ such that there does not exist lines ℓ_1, ℓ_2, ℓ_3 meeting a point with the angles between different lines all the same (and thus equal to $\pi/3$) such that the ℓ_i divide P into 6 regions of equal area. Hint : Make P long and skinny.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic continuous piecewise linear function. Prove that f has inscribed chords of any length. Hint : For some α, define $g_\alpha : \mathbb{R} \to \mathbb{R}$ by $g_\alpha(x) = f(x+\alpha) - f(x)$. Prove that if g_α is never 0, then $f(x)$ goes to either ∞ or $-\infty$ as x goes to ∞. Why is this a contradiction?

4. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1) = 0$. Prove that there is some $\epsilon > 0$ such that $D(f)$ contains $[0,\epsilon]$. Hint : The function f has to have global maxima and minima. Moreover, unless f is constant (in which case the problem is trivial), one of these has to occur at some point $a \in (0,1)$. Choose $\epsilon > 0$ such that $(a-\epsilon, a+\epsilon) \subset (0,1)$, and consider $c \in [0,\epsilon]$. Define $g(x) = f(x) - f(x+c)$. Finding a chord of length c is equivalent to finding a zero of $g(x)$. Use the intermediate value theorem to show that $g(x)$ has to have a zero somewhere.

   REMARK : In a previous version of this problem set, I did not have a hint but I assumed that $f(x)$ was piecewise-linear. The above hint does not use piecewise-linearity, but if you want to prove it a different way then you are free to assume that $f(x)$ is piecewise-linear (you will still get full credit).

5. Let P_1 and P_2 be disjoint convex polygons in $\mathbb{R}^2$. Prove that there exists a line ℓ that simultaneously divides P_1 into two pieces with equal areas and P_2 into two pieces with equal areas. Hint : For each angle $\theta \in [0,\pi]$, we already proved that we can find a unique line $\ell(\theta)$ with the following properties.

   - $\ell(\theta)$ divides P_1 into two parts with equal areas.
   - If $\theta = 0$ or $\theta = \pi$, then $\ell(\theta)$ is horizontal. If instead $\theta \in (0,\pi)$, then $\ell(\theta)$ makes an angle of θ with the x-axis.

   Let θ vary, and prove using the intermediate value theorem that we can choose θ such that $\ell(\theta)$ also divides P_2 in half.