Math 366 : Geometry
Problem Set 7

1. The *regular cube* in $\mathbb{R}^3$ is the convex hull C of the 8 points $(\pm 1, \pm 1, \pm 1)$. Prove that C is a regular polytope. Hint : Imitate the proof I gave in class for the regularity of the icosahedron.

2. The *regular tetrahedron* in $\mathbb{R}^3$ is the convex hull T of four points $x_1, \ldots, x_4 \in \mathbb{R}^3$ such that $\text{dist}(x_i, x_j) = 1$ for all $i \neq j$.

   (a) Note that it is not completely obvious that such points exist. Find them.

   (b) Prove that T is a regular polytope. Hint : Imitate the proof I gave in class for the regularity of the icosahedron.

3. Recall that the *regular icosahedron* $\mathcal{I}$ is the convex hull of the 12 points $(\pm 1, \pm\phi, 0)$ and $(\pm\phi, 0, \pm 1)$ and $(0, \pm 1, \pm\phi)$, where ϕ is the golden mean $\frac{1+\sqrt{5}}{2}$.

   (a) Let ℓ be the line through the points $(1, \phi, 0)$ and $(-1, -\phi, 0)$ and let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation about ℓ by an angle $2\pi/5$. Prove that f restricts to a symmetry of $\mathcal{I}$. Hint : Determine a formula for f and then check that f takes vertices to vertices.

   (b) Let P be the plane containing the points $(1, \phi, 0)$ and $(-1, -\phi, 0)$ and $(\phi, 0, 1)$ and let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection in P. Prove that g restricts to a symmetry of $\mathcal{I}$.