Math 366 : Geometry
Problem Set 8

1. Fix some $c \geq 0$ and $d > 0$. Let $E \subset \mathbb{R}^2$ be the set of points (x, y) such that $\text{dist}((x, y), (c, 0)) + \text{dist}((x, y), (-c, 0)) = d$. Show that for some $a, b > 0$ the set E can also be described as the set of points satisfying $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (the standard form for an ellipse).

2. Let $Q \subset \mathbb{R}^2$ be a convex polygon and let e be an edge of Q of maximal length (there might exist more than one such edge). Prove that there exists a vertex of Q that is not an endpoint of e and which projects orthogonally onto e. Hint : Let the endpoints of e be x_1 and x_2, and let ℓ_i be the straight line through x_i which is perpendicular to e. Prove that if there is no vertex as in the problem, then there is an edge e' that intersects both ℓ_1 and ℓ_2 in its interior. Why does this lead to a contradiction?

3. Let $C \subset \mathbb{R}^2$ be a smooth simple closed curve. Prove that there exist three distinct points $x_1, x_2, x_3 \in C$ such that the following holds for all $1 \leq i \leq 3$. Let j and k be the other numbers between 1 and 3. Then the tangent line to C at x_i is parallel to the line segment from x_j to x_k. Hint : Let $x_1, x_2, x_3 \in C$ be the points forming a triangle of greatest possible area. Prove that the area assumption implies that the point x_i is the point in C whose distance from the segment from x_j to x_k is maximal. Use this (and calculus) to prove that the tangent line to C at x_i is parallel to the line segment from x_j to x_k.

4. Let $S = [0, 1] \times [0, 1]$ be the unit square. Give explicit examples of infinitely many irreducible periodic billiard trajectories in S (of course, such trajectories cannot pass through the vertices).

5. Let $U \subset \mathbb{R}^2$ be a convex region with smooth boundary and let $x, y \in U$ be distinct. Then there exists infinitely many distinct billiard trajectories starting at x and ending at y. Hint : Imitate the proof of Birkhoff's theorem giving infinitely many distinct irreducible periodic billiard trajectories.