Math 366 : Geometry
Midterm

This is a pledged midterm exam. It is due on Monday, March 11 at the start of class. There is no time limit. You are allowed to use the textbook, your notes, other books, and static internet resources like wikipedia; however, you are not allowed to talk to each other (or anyone else) or to use internet question/answer sites like `math.stackexchange.com` or `mathoverflow.net`.

1. Construct a convex set $U \subset \mathbb{R}^2$ and a continuous function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(U)$ is not convex. Hint : You can take U to be a line segment.

2. Let $U_1 \subset U_2 \subset U_3 \subset \cdots$ be an increasing sequence of convex subsets of $\mathbb{R}^n$. Prove that $\cup_{i=1}^{\infty} U_i$ is convex.

3. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ be convex sets. Define $W = \{\vec{u} + \vec{v} \mid \vec{u} \in U \text{ and } \vec{v} \in V\}$. Prove that W is convex.

4. Prove that the Sylvexter-Gallai theorem does *not* hold for infinite sets. In other words, construct (with proof!) an infinite set $U \subset \mathbb{R}^2$ such that if a line ℓ passes through two distinct points $x, y \in U$, then ℓ passes through a third point of U.

5. Let $P \subset \mathbb{R}^2$ be a convex polygon. Prove that there exists a line ℓ in $\mathbb{R}^2$ that divides P into two convex polygons whose perimeters are equal.

6. Recall that we proved in class that every finite tree T had a leaf, that is, a vertex v_0 such that v_0 is the endpoint of precisely one edge of T. Does this hold for infinite trees? Either prove it or construct a counterexample.