Math 444/539, Final Exam

You have five hours to take this midterm exam. You are allowed to consult your course notes, the notes posted on the course webpage, and Hatcher's book. No other sources may be consulted. This exam is due Wednesday December 16 in my office.

1. Let F_n be a free group of rank n. For each $n \geq 2$, prove that F_2 contains a finite-index normal subgroup isomorphic to F_n.

2. Let $T = S^1 \times S^1$ and let $f : T \to T$ be defined by $f(x, y) = (2x+y, x+y)$. Here we are viewing S^1 as $\mathbb{R}/\mathbb{Z}$. Let $X = (T \times [0, 1])/\sim$ be the 3-manifold obtain by identifying $(x, y) \times \{0\}$ with $f(x, y) \times \{1\}$. Compute $\pi_1(X)$.

3. Let $L_1, \ldots, L_n$ be disjoint straight lines in $\mathbb{R}^3$. Calculate $\pi_1(\mathbb{R}^3 \setminus (L_1 \cup \cdots \cup L_n))$.

4. Let $W = S^1 \vee S^1$. Construct three connected 4-fold covers of W that are distinct up to covering space equivalence, including at least 1 irregular cover. For each of these three covers, describe the covering map, say whether or not the cover is regular, and give the corresponding subgroup of $\pi_1(W)$.

5. Let X be a connected CW complex such that $\pi_1(X)$ is a finite group. Prove that any continuous map $f : X \to S^1$ is homotopic to a constant map.

6. Let $\{p_1, \ldots, p_k\}$ be a set of k distinct points in S^2. Let X be the quotient space of S^2 that results from identifying all the p_i to a single point.

   (a) Construct a CW complex structure on X.

   (b) Calculate $\pi_1(X)$.

7. Let X and Y be compact, connected, oriented n-manifolds and let $f : X \to Y$ be a map of degree 1. Prove that the induced map $f_* : \pi_1(X) \to \pi_1(Y)$ is surjective. Hint: this uses covering spaces.