Math 444/539, Homework 1

1. Recall that $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^1 + \cdots + x_{n+1}^2 = 1\}$. Define a function $f : S^n \to \mathbb{R}$ via the formula
$$f(x_1, \ldots, x_{n+1}) = \frac{x_{n+1}^2}{7 + e^{x_1}}.$$
Prove that f is smooth directly (that is, by determining its behavior on the charts in an atlas).

2. Define two different smooth atlases on $\mathbb{R}$:

   - The atlas $\mathcal{A}$ has a single chart $\phi : U \to V$ with $U = V = \mathbb{R}$ and $\phi(x) = x$.
   - The atlas $\mathcal{A}$ has a single chart $\phi' : U' \to V'$ with $U' = V' = \mathbb{R}$ and $\phi'(x) = x^3$.

   This gives two different smooth manifolds $(\mathbb{R}, \mathcal{A})$ and $(\mathbb{R}, \mathcal{A}')$ whose underlying set is $\mathbb{R}$. Prove the following things:

   (a) The identity map $i : \mathbb{R} \to \mathbb{R}$ is a smooth homeomorphism from $(\mathbb{R}, \mathcal{A})$ to $(\mathbb{R}, \mathcal{A}')$, but is not a diffeomorphism (here recall that a diffeomorphism is a smooth homeomorphism whose inverse is also smooth).

   (b) Prove that exists some $j : \mathbb{R} \to \mathbb{R}$ which is a smooth diffeomorphism from $(\mathbb{R}, \mathcal{A})$ to $(\mathbb{R}, \mathcal{A}')$.

   In fact, one can prove that any two smooth atlases on $\mathbb{R}$ yields diffeomorphic smooth manifolds. Even more is true: for any manifold of dimension at most 3, any two smooth atlases yield diffeomorphic smooth manifolds (one says that there are no "exotic" smooth structures in these dimensions). Remarkably, this starts to fail in dimension 4; in fact, there exist uncountably many non-diffeomorphic smooth structures on $\mathbb{R}^4$.

3. Recall from class the construction of the tangent bundle of a smooth manifolds M^n:

   - Let $\mathcal{A} = \{\phi_i : U_i \to V_i\}_{i \in I}$ be an atlas on M^n. For all $i, j \in I$, let $\tau_{ij} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ be the transition function for $\mathcal{A}$, i.e. the function $\tau_{ij} = \phi_j \circ \phi_i^{-1}$. By definition, each τ_{ij} is a smooth function (its domain and codomain are both open subsets of $\mathbb{R}^n$, so the notion of "smooth" is the one from multivariable calculus). Letting $TV_i = V_i \times \mathbb{R}^n$ be the tangent bundle of V_i, we then set
   $$TM^n = \bigsqcup_{i \in I} TV_i / \sim,$$
   where $\sim$ identifies for all $i, j \in I$ the sets
   $$T\phi_i(U_i \cap U_j) \subset TV_i \quad \text{and} \quad T\phi_j(U_i \cap U_j) \subset TV_j$$
   via the derivative $D\tau_{ij}$ of the transition map. Remark: We allow $i = j$, in which case the transition function goes from $\phi_i(U_i \cap U_i) = \phi_i(U_i) = V_i$ to itself via the identity.

   Prove the following things about TM^n:

   (a) Prove that if $a_1 \sim a_2 \sim \cdots \sim a_k$, then $a_1 \sim a_k$. Here the a_j are points in the various TV_i.

   (b) For all $q \in V_i$, the map $T_q V_i \to TM^n$ is injective; here recall that $T_q V_i = \{q\} \times \mathbb{R}^n$. Setting $p = \phi_i^{-1}(q)$, by definition we have that $T_p M^n$ is the image of this map you just proved is injective.

(c) Consider $p \in M^n$ and $i, j \in I$ such that $p \in U_i \cap U_j$. Setting $q_i = \phi_i(p)$ and $q_j = \phi_j(p)$, the previous step identifies $T_p M^n$ with both $T_{q_i} V_i$ and $T_{q_j} V_j$. Both $T_{q_i} V_i$ and $T_{q_j} V_j$ are n-dimensional vector spaces over $\mathbb{R}$, so this gives two different vector space structures on $T_p M^n$. Prove that these vector space structures are the same, i.e. that they induce the same operations of addition and scalar multiplication. We conclude that $T_p M^n$ is in a natural way a vector space.

4. (a) Prove that the inclusion map $S^n \hookrightarrow \mathbb{R}^{n+1}$ is an embedding (the most important thing to check is that the induced map on tangent spaces in injective).

   (b) Prove that under the embedding $S^n \hookrightarrow \mathbb{R}^{n+1}$, the image of the tangent bundle TS^n consists of

$$\{(x, \vec{v}) \in S^n \times \mathbb{R}^{n+1} \mid \text{the vector from 0 to } x \text{ is orthogonal to } \vec{v}\} \subset T\mathbb{R}^{n+1} = \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}.$$

   Of course, this is the tangent bundle to S^n you learned about in multivariable calculus!

5. Recall from the lectures the beginning of the proof that if M^n is a compact n-manifold, then for some $m \gg 0$ there exists an embedding $f : M^n \to \mathbb{R}^m$.

   - Since M^n is compact, there exists a finite atlas

$$\mathcal{A} = \{\phi_i : U_i \to V_i\}_{i=1}^{\ell}.$$

   Choose open subsets $W_i \subset U_i$ such that $\{W_i\}_{i=1}^{\ell}$ is still a cover of M^n and such that the closure of W_i in U_i is compact. Using standard multivariable calculus tools, we can then construct a *bump function* for $\phi_i(W_i) \subset V_i$, i.e. a smooth function $\zeta_i : V_i \to \mathbb{R}$ such that $\zeta_i|_{\phi_i(W_i)} = 1$ and such that the closure of the set $\{x \in V_i \mid \zeta_i(x) \neq 0\}$ is compact. Define $\nu_i : M^n \to \mathbb{R}$ via the formula

$$\nu_i(p) = \begin{cases} \zeta_i(\phi_i(p)) & \text{if } p \in U_i, \\ 0 & \text{otherwise.} \end{cases}$$

   Clearly ν_i is a smooth function. Next, define a function $\eta_i : M^n \to \mathbb{R}^n$ via the formula

$$\eta_i(p) = \begin{cases} \nu_i(p) \cdot \phi_i(p) & \text{if } p \in U_i, \\ 0 & \text{otherwise.} \end{cases}$$

   Again, η_i is a smooth function. Finally, define $f : M^n \to \mathbb{R}^{\ell(n+1)}$ via the formula

$$f(p) = (\nu_i(p), \eta_1(p), \ldots, \nu_\ell(p), \eta_\ell(p)).$$

   The function f is then a smooth map.

Problem: Prove that f is an embedding. Again, the most important thing to check is that the induced map on tangent spaces is injective.