Math 444/539, Homework 10

1. Let X be the wedge of two circles with wedge point x_0 and let $a, b \in \pi_1(X, x_0)$ be the loops around the circles, so $\pi_1(X, x_0) = \langle a, b \mid \rangle$. Construct the cover of X corresponding to the subgroup *normally* generated by a^2 and b^2 and $(ab)^4$. The word "normally" indicates that this group is generated by conjugates of the indicated elements.

2. Find all the connected covering spaces of $\mathbb{RP}^2 \vee \mathbb{RP}^2$.

3. Let $f : Y \to X$ be a simply-connected covering space of X, let $A \subset X$ be a path-connected, locally path-connected subspace, and let $B \subset Y$ be a path component of $f^{-1}(A)$. Prove that $f|_B : B \to A$ is the covering space corresponding to the kernel of the map $\pi_1(A) \to \pi_1(X)$.

4. Consider a group G acting freely on a Hausdorff space X. Assume that for each $x \in X$, there exists an neighborhood U of x such that the set $\{g \in G \mid gU \cap U \neq \emptyset\}$ is finite. Prove that the action of G is a covering space action.

5. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $\phi(x, y) = (2x, y/2)$. This generates an action of $\mathbb{Z}$ on $X = \mathbb{R}^2 \setminus \{0\}$. Show this action is a covering space action and compute $\pi_1(X/\mathbb{Z})$. Show the orbit space $X/\mathbb{Z}$ is non-Hausdorff, and describe how it is a union of four subspaces homeomorphic to $S^1 \times \mathbb{R}$, coming from the complementary components of the x-axis and the y-axis.

6. Consider covering spaces $f : Y \to X$ with Y and X connected CW complexes, the cells of Y projecting homeomorphically onto cells of X. Restricting f to the 1-skeleton then gives a covering space $Y^{(1)} \to X^{(1)}$ over the 1-skeleton of X. Prove the following.

    (a) Two such covering spaces $Y_1 \to X$ and $Y_2 \to X$ are isomorphic iff the restrictions $(Y_1)^{(1)} \to X^{(1)}$ and $(Y_2)^{(1)} \to X^{(1)}$ are isomorphic.

    (b) $Y \to X$ is a regular covering space iff $Y^{(1)} \to X^{(1)}$ is a regular covering space.

    (c) The groups of deck transformations of the coverings $Y \to X$ and $Y^{(1)} \to X^{(1)}$ are isomorphic, via the restriction map.