Math 444/539, Homework 3

1. Let M^n be a smooth n-manifold embedded in $\mathbb{R}^m$. Prove that the normal bundle $N_{\mathbb{R}^m/M^n}$ is a smooth m-dimensional manifold.

2. Let $f : \mathbb{R}^m \to S^{m-n}$ be a smooth map and let $p \in S^{m-n}$ be a regular value of f. Set $M^n = f^{-1}(p)$. Prove that the normal bundle $N_{\mathbb{R}^m/M^n}$ can be decomposed as $M^n \times \mathbb{R}^{m-n}$.

3. Let M^n be a smooth compact n-manifold embedded in $\mathbb{R}^m$. Assume that the normal bundle splits as $N_{\mathbb{R}^m/M^n} = M^n \times \mathbb{R}^{m-n}$. Construct a smooth map $f : \mathbb{R}^m \to S^{m-n}$ and a regular value $p \in S^{m-n}$ such that $f^{-1}(p) = M^n$. Hint: Use the tubular neighborhood theorem in the strong form we proved in class involving the normal bundle and regard S^{m-n} as $\mathbb{R}^{m-n}$ together with a point at ∞.

4. Prove that $M \times N$ is orientable if and only if both M and N are orientable.

5. Let M^n be a smooth compact oriented n-manifold with boundary and let $f : M^n \to S^{n-1}$ be a smooth map. Prove that the degree of $f|_{\partial M^n}$ is 0.

6. Let $M_1^{n_1}$ and $M_2^{n_2}$ be disjoint smooth oriented submanifolds of $\mathbb{R}^{n_1+n_2+1}$. The *linking number* of $M_1^{n_1}$ and $M_2^{n_2}$, denoted $\mathrm{lk}(M_1^{n_1}, M_2^{n_2})$, is the degree of the smooth map $\phi : M_1^{n_1} \times M_2^{n_2} \to S^{n_1+n_2}$ defined via the formula
$$\phi(x, y) = \frac{x - y}{\|x - y\|}.$$
Prove that
$$\mathrm{lk}(M_1^{n_1}, M_2^{n_2}) = (-1)^{(n_1+1)(n_2+1)}\mathrm{lk}(M_2^{n_2}, M_1^{n_1}).$$

7. Construct (with proof) two disjoint circles X and Y embedded in $\mathbb{R}^3$ such that $\mathrm{lk}(X, Y) = 1$.

8. Let $(\vec{v}_1, \ldots, \vec{v}_{n+1})$ be a basis for $\mathbb{R}^{n+1}$ with the following properties.
   - $(\vec{v}_1, \ldots, \vec{v}_n)$ is a basis for $\mathbb{R}^n \subset \mathbb{R}^{n+1}$.
   - $(\vec{v}_1, \ldots, \vec{v}_{n+1})$ induces the standard orientation on $\mathbb{R}^{n+1}$.
   - The $(n+1)^{\text{st}}$ coordinate of $\vec{v}_{n+1}$ is positive.

   Let b be the orientation on $\mathbb{R}^n$ induced by the basis $(\vec{v}_1, \ldots, \vec{v}_n)$. Problem: prove that b is independent of the basis $(\vec{v}_1, \ldots, \vec{v}_{n+1})$ (among bases satisfying the above three properties).