Math 444/539, Homework 5

1. Let X be a path-connected topological space with **abelian** fundamental group. Fix two points $p, q \in X$. Recall that $\varphi_\gamma : \pi_1(X, q) \to \pi_1(X, p)$ is the homomorphism associated to an equivalence class γ of paths from p to q. Prove that if γ and γ' are two paths from p to q, then $\varphi_\gamma = \varphi_{\gamma'}$.

2. Let X be a topological space, let $p, q \in X$ be two points, and let f and g be two paths from p to q. Prove that f is equivalent to g if and only if $f \cdot \overline{g}$ is equivalent to the constant path e_p.

3. Let X be a topological space. Prove that the following three conditions are equivalent.

   (a) Every map $S^1 \to X$ is homotopic to a constant map.

   (b) For every map $f : S^1 \to X$, there exists a map $g : D^2 \to X$ such that $g|_{\partial D^2} = f$.

   (c) For all $p \in X$, we have $\pi_1(X, p) = 1$.

   I want the emphasize that in this problem, "homotopic" means "homotopic without regards to basepoints".

4. Let G be a topological group. Let $e \in G$ be the identity element. Prove that $\pi_1(G, e)$ is abelian. Hint : in addition to the multiplication of loops $\cdot$ in $\pi_1(G, e)$, the group structure of G gives another way of multiplying loops. Namely, for loops f and g based at e, we can define $f * g$ to be the loop $t \mapsto f(t)g(t)$. The first step is to prove that the loop $f * g$ is equivalent to the loop $g \cdot f$.

5. Let X be a topological space and let $\{U_\alpha\}$ be an open covering of X with the following properties.

   (a) There exists a point $p \in X$ such that $p \in U_\alpha$ for all α.

   (b) Each U_α is *simply-connected*, that is, U_α is path-connected and $\pi_1(U_\alpha, q) = 1$ for all $q \in U_\alpha$.

   (c) For $\alpha \neq \beta$, the set $U_\alpha \cap U_\beta$ is path-connected.

   Prove that X is simply-connected. Hint : consider $\gamma \in \pi_1(X, p)$. Prove that we can write $\gamma = \gamma_1 \cdots \gamma_k$, where $\gamma_i \in \pi_1(X, p)$ can be realized by a loop based at p that lies entirely inside one of the U_α. The notion of the *Lebesgue number* of a covering from point-set topology will be useful here.

6. Using the previous problem, prove that S^n is simply-connected for $n \geq 2$.

7. Consider a map $f : S^1 \to S^1$. Pick some path γ from $f(1) \in S^1$ to $1 \in S^1$. We therefore get an induced sequence of maps

   $$\mathbb{Z} = \pi_1(S^1, 1) \xrightarrow{f_*} \pi_1(S^1, f(1)) \xrightarrow{\phi_\gamma} \pi_1(S^1, 1) = \mathbb{Z}.$$

   which we will denote $\psi : \mathbb{Z} \to \mathbb{Z}$.

   (a) Prove that ψ is multiplication by some integer n.

   (b) Prove that n is independent of the choice of path γ.

   (c) Prove that n is the degree of the map f.

8. Prove that if $f : S^1 \to S^1$ has degree different from 1, then there exists some $x \in S^1$ such that $f(x) = x$.