Math 444/539, Homework 6

1. Let S be a set. Prove that every word in $S^{\pm 1}$ is equivalent to a unique reduced word. Hint: Let $\mathcal{W}$ be the set of reduced words on $S^{\pm 1}$. For $s \in S^{\pm 1}$, define a function $\phi_s : \mathcal{W} \to \mathcal{W}$ by defining $\phi_s(w)$ to be sw if w does not begin with s^{-1} and to equal v if w begins with s^{-1} and $w = s^{-1}v$. Show that $\phi_s \circ \phi_{s^{-1}} = \mathrm{id}_{\mathcal{W}}$ for all $s \in S^{\pm 1}$. Next, if $w = s_1^{\epsilon_1} \cdots s_k^{\epsilon_k}$ with $s_i \in S$ and $\epsilon = \pm 1$ is an arbitrary word on $S^{\pm 1}$, then define $\phi_w : \mathcal{W} \to \mathcal{W}$ via the formula
$$\phi_w = \phi_{s_1^{\epsilon_1}} \circ \cdots \circ \phi_{s_k^{\epsilon_k}}.$$
Show that if w and w' are equivalent words on $S^{\pm 1}$, then $\phi_w = \phi_{w'}$. Use this to deduce the desired result.

2. If S and S' are finite sets of different cardinality, prove that $F(S)$ is not isomorphic to $F(S')$.

3. Prove directly from the definition of a free group and a free product (in terms of the universal mapping property) that $\mathbb{Z} * \mathbb{Z}$ is isomorphic to a free group on a 2-element set.

4. Prove that the symmetric group S_n on n letters $\{1, \ldots, n\}$ has the presentation
$$\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i^2 = 1 \text{ for } 1 \le i \le n-1,\ [\sigma_i, \sigma_j] = 1 \text{ if } 1 \le i, j \le n-1 \text{ satisfy } |i-j| > 1, \text{ and } \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } 1 \le i \le n-2 \rangle.$$
Here σ_i corresponds to the transposition $(i, i+1)$. Hint: Prove this by induction on n.

5. Let G be the set of all matrices of the form $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ with $a, b, c \in \mathbb{Z}$. Prove that G is a group and that G can be given the presentation $\langle x,\, y,\, z \mid [x, y] = z,\, [x, z] = 1,\, [y, z] = 1 \rangle$. Hint: the generator x corresponds to the matrix $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, the element y corresponds to the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, and the element z corresponds to the matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

6. Define $G = \langle a,\, b \mid a^2 = b^3 \rangle$. Prove that G is not abelian. Hint: try to find a surjective homomorphism from G to the symmetric group S_3.

7. Let G and H be nontrivial groups. Prove that $G * H$ has a trivial center and that if $x \in G * H$ satisfies $x^n = 1$ for some $n \ge 1$, then x is conjugate to an element of either G or H.