Math 444/539, Homework 7

1. Let G and H be nontrivial groups. Prove that $G * H$ has a trivial center and that if $x \in G * H$ satisfies $x^n = 1$ for some $n \geq 1$, then x is conjugate to an element of either G or H.

2. Let $X \subset \mathbb{R}^n$ be a finite set of points. Assume that $n \geq 3$. Prove that $\pi_1(\mathbb{R}^n \setminus X) = 1$.

3. Let $X \subset \mathbb{R}^3$ be a set of n distinct lines through the origin. Calculate $\pi_1(\mathbb{R}^3 \setminus X)$.

4. Let X equal $T^2 \sqcup T^2$ modulo the equivalence relation that identifies the circles $S^1 \times 1$ in the two tori homeomorphically. Calculate $\pi_1(X)$.

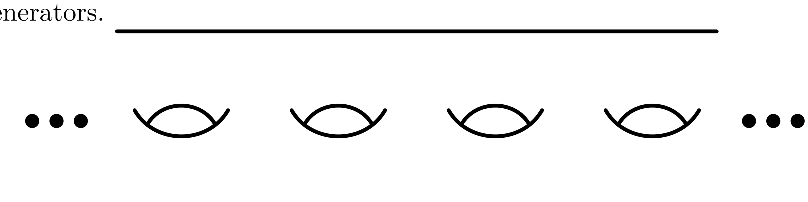
5. Let $X = \cup_{n=1}^{\infty} X_n$, where $X_n \subset \mathbb{R}^2$ is the circle of center $(1/n, 0)$ and radius $1/n$. Let $p = (0, 0)$. Prove that $\pi_1(X, p)$ is uncountable. Hint : construct a retraction $r_n : X \to X_n$, and thus a surjection $(r_n)_* : \pi_1(X, p) \to \pi_1(X_n, p) = \mathbb{Z}$. Combine the r_n together to get a map $R : \pi_1(X, p) \to \prod_{n=1}^{\infty} \pi_1(X_n, p)$. Prove that R is surjective.

6. Let T^2 be the 2-torus. Recall that $\pi_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$. Consider $(n, m) \in \mathbb{Z} \oplus \mathbb{Z}$. Assume that n and m are relatively prime. Prove that the curve on T^2 representing the homotopy class of (n, m) can be chosen so that it has no self-intersections. Hint: Use the projection $\mathbb{R} \to S^1$ that was used to calculate $\pi_1(S^1)$ to construct a projection $\rho : \mathbb{R}^2 \to T^2$. The curve you want will be the image of a straight line in $\mathbb{R}^2$.

7. Let $\Sigma_{g,n}$ be the result of removing n disjoint open discs from an oriented genus g surface. Thus $\Sigma_{g,n}$ is a compact manifold with boundary whose boundary consists of n circles. Assume that $g \geq 2$ and that $n \geq 1$. Prove that $\pi_1(\Sigma_{g,n})$ is a free group on $2g + n - 1$ generators. You can use the fact that the diffeomorphism type of this surface does not depend on which discs you remove.

8. Let Σ_g be an oriented genus g surface. Assume that $g \geq 2$. Prove that $\pi_1(\Sigma_g)$ is not abelian. Hint : find a surjective homomorphism from $\pi_1(\Sigma_g)$ to the dihedral group of order 8.

9. Prove that the fundamental group of the following noncompact surface is free on infinitely many generators.



10. Let $f : T^2 \to T^2$ be a map satisfying $f(p) = p$ for some point p. Since $\pi_1(T^2, p) \cong \mathbb{Z}^2$, we get an induced map $f_* : \mathbb{Z}^2 \to \mathbb{Z}^2$; ie a 2×2 integer matrix. Define M_f to be $T^2 \times I$ modulo the equivalence relation that identifies $(x, 1)$ with $(f(x), 0)$ (this is called the mapping torus of f). Compute $\pi_1(M_f)$ in terms of the above matrix.