

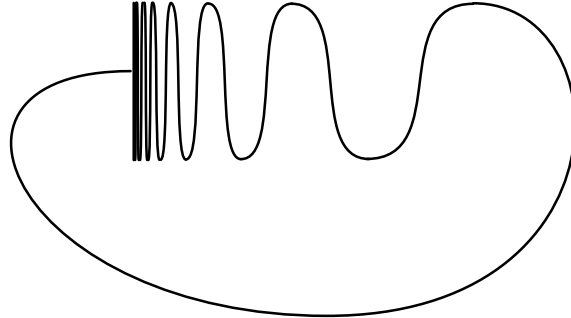
Math 444/539 : Geometric Topology

Problem Set 9

1. For a covering space $\rho : \tilde{X} \rightarrow X$ and a subspace $A \subset X$, let $\tilde{A} = \rho^{-1}(A)$. Prove that $\rho|_{\tilde{A}} : \tilde{A} \rightarrow A$ is a covering space.
2. Show that if $\rho_1 : \tilde{X}_1 \rightarrow X_1$ and $\rho_2 : \tilde{X}_2 \rightarrow X_2$ are covering spaces, then so is $\rho_1 \times \rho_2 : \tilde{X}_1 \times \tilde{X}_2 \rightarrow X_1 \times X_2$.
3. Let $\rho : \tilde{X} \rightarrow X$ be a covering space with $\rho^{-1}(x)$ finite and nonempty for all $x \in X$. Prove that \tilde{X} is compact Hausdorff if and only if X is compact Hausdorff.
4. Construct the universal cover of the following space:

$$X = \{x \in \mathbb{R}^3 \mid \|x\| = 1\} \cup \{(x, 0, 0) \in \mathbb{R}^3 \mid -1 \leq x \leq 1\}.$$

5. Let X be the subspace of \mathbb{R}^2 consisting of the four sides of the square $[0, 1] \times [0, 1]$ together with the segments of the vertical lines $x = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ inside the square. Show that for every covering space $\rho : \tilde{X} \rightarrow X$, there is some neighborhood of the left edge of X that lifts homeomorphically to \tilde{X} . Deduce that X has no simply-connected covering space.
6. Let Y be the *quasi-circle* shown in the following figure:



Thus Y is a closed subspace of \mathbb{R}^2 consisting of the union of the set

$$\{(x, y) \in \mathbb{R}^2 \mid 0 < x \leq 1, y = \sin(\frac{1}{x})\},$$

the set

$$\{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\},$$

and an arc joining the point $(0, 0)$ to the point $(1, \sin(1))$. Collapsing the segment of Y in the y -axis to a point gives a quotient map $f : Y \rightarrow S^1$. Show that f does not lift to the universal covering space $\rho : \mathbb{R} \rightarrow S^1$ even though $\pi_1(Y) = 0$. Thus local path-connectedness of Y is a necessary hypothesis in the lifting criterion.

7. Let X be a path-connected, locally path-connected space with $\pi_1(X)$ a finite group. Prove that every map $f : X \rightarrow S^1$ is nullhomotopic.
8. Find all connected 2-sheeted and 3-sheeted coverings of the wedge of two circles (up to isomorphism of covering spaces).