Math 444/539, Midterm Exam

You have unlimited time to take this midterm exam. You are allowed to consult your course notes and the notes posted on the course webpage. No other sources may be consulted. This exam is due Thursday October 22 in class.

1. Define $\mathbb{CP}^n$ to be the set of equivalence classes of $\mathbb{C}^{n+1} \setminus \{0\}$ under the equivalence relation that identifies x and λx for all $x \in \mathbb{C}^{n+1} \setminus \{0\}$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Endow $\mathbb{CP}^n$ with the quotient topology, so a set $U \subset \mathbb{CP}^n$ is open if and only if the preimage of U in $\mathbb{C}^{n+1} \setminus \{0\}$ under the natural surjection $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is open. Problem: prove that $\mathbb{CP}^n$ is a smooth orientable manifold of dimension $2n$.

2. Let $f : M_1^{n_1} \to M_2^{n_2}$ be a smooth map between smooth manifolds. Also, let $i : X^k \hookrightarrow M_2^{n_2}$ be an embedding of a smooth manifold X^k into $M_2^{n_2}$. Assume that for all $p \in M_1^{n_1}$ and $x \in X^k$ with $f(p) = i(x)$, the span of the images of $D_p f : T_p M_1^{n_1} \to T_{f(p)} M_2^{n_2}$ and $D_x i : T_x X^k \to T_{i(x)} M_2^{n_2}$ span $T_{f(p)} M_2^{n_2} = T_{i(x)} M_2^{n_2}$. Prove that the set $f^{-1}(i(X^k)) \subset M_1^{n_1}$ is a smooth $(n_1 - (n_2 - k))$-dimensional smooth submanifold of $M_1^{n_1}$. Hint: One special case of this is where X^k is a single point x and $i(x)$ is a regular value of f. Use the local immersion theorem to reduce this (locally) to this special case.

3. Fix some real numbers $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n+1}$. Regarding S^n as a subspace of $\mathbb{R}^{n+1}$, define a map $f : S^n \to \mathbb{R}$ via the formula
$$f(x_1, \ldots, x_{n+1}) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_{n+1} x_{n+1}^2 \qquad \text{for } (x_1, \ldots, x_{n+1}) \in S^n \subset \mathbb{R}^{n+1}.$$
   (a) Prove that the regular values of f are exactly the set $\mathbb{R} \setminus \{\lambda_1, \ldots, \lambda_{n+1}\}$.
   (b) Consider $a \in \mathbb{R}$ such that $\lambda_k < a < \lambda_{k+1}$ for some $1 \leq k \leq n$. Define $X = f^{-1}(a)$. Prove that X is diffeomorphic to $S^{k-1} \times S^{n-k}$.

4. Let M^n be a smooth connected manifold and let $O(M^n)$ be the set of pairs (p, λ), where p is a point of M^n and λ is an orientation on $T_p M^n$. Let $\pi : O(M^n) \to M^n$ be the projection onto the first coordinate.
   (a) Construct a topology on $O(M^n)$ such that $O(M^n)$ is a smooth n-dimensional manifold and π is a local diffeomorphism.
   (b) Prove that $O(M^n)$ is orientable.
   (c) Prove that M^n is orientable if and only if $O(M^n)$ has two connected components.
   (d) Let B be the 2-dimensional Mobius strip, which we define to be the quotient of $[-1, 1] \times (0, 1)$ that identifies $(t, 0)$ and $(-t, 1)$ for all $t \in [-1, 1]$. Identify $O(B)$ and use this to prove that B is not orientable.