Math 60330: Basic Geometry and Topology
Problem Set 1

1. Let (X, d_X) and (Y, d_Y) be metric spaces and let $f\colon X \to Y$ be a function. Prove that the following two definitions of f being continuous are equivalent.

   - The function f is continuous if for all open sets $U \subset Y$, the preimage $f^{-1}(U) \subset X$ is open.
   - The function f is continuous if for all $\epsilon > 0$ and all $p \in X$, there exists some $\delta > 0$ such that if $q \in X$ satisfy $d_X(p,q) < \delta$, then $d_Y(f(p), f(q)) < \epsilon$.

2. (a) Let $\sim$ be the following equivalence relation on $\mathbb{R}^2$:
   $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 + y_1^2 = x_2 + y_2^2$.
   Give $\mathbb{R}^2/\sim$ the quotient topology coming from the projection $\mathbb{R}^2 \to \mathbb{R}^2/\sim$. The space $\mathbb{R}^2$ by $\sim$ is a familiar one. What space is it?

   (b) Repeat part a for the following equivalence relation:
   $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1^2 + y_1^2 = x_2^2 + y_2^2$.

3. Let $\sim$ be the following equivalence relation on $X = [-1, 1] \times \mathbb{R} \subset \mathbb{R}^2$:

   $(x_1, y_1) \sim (x_2, y_2)$ if and only if one of the following hold:

   - $x_1 = x_2 = 1$, or
   - $x_1 = x_2 = -1$, or
   - $-1 < x_1, x_2 < 1$ and $(x_1^2 - 1)e^{y_1} = (x_2^2 - 1)e^{y_2}$.

   (a) Draw the equivalence classes for $\sim$.

   (b) Prove that the quotient of X by $\sim$ is not Hausdorff.

4. Let X be a CW complex.

   (a) Prove that if X has finitely many cells, then X is compact.

   (b) Let $C \subset X$ be a compact subset (not necessarily a subcomplex). Prove that C only intersects finitely many cells of X.

5. Construct CW complex structures on the following spaces.

   (a) An n-dimensional torus.

   (b) Letting $\{p_1, \ldots, p_n\}$ be n distinct points on S^2, the quotient space of S^2 that identifies all the p_i to a single point.