Math 60330: Basic Geometry and Topology
Problem Set 2

1. (a) Carefully prove that the following are covering spaces (in particular, give explicit trivializing neighborhoods for an arbitrary point in their base). Recall that $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$.

   i. The map $\pi\colon \mathbb{C} \to \mathbb{C}^*$ defined by $\pi(z) = e^z$.

   ii. For $n \in \mathbb{Z}\backslash\{0\}$, the map $\pi\colon \mathbb{C}^* \to \mathbb{C}^*$ defined by $\pi(z) = z^n$.

   (b) Prove that the map $\pi\colon \mathbb{C} \to \mathbb{C}$ defined by $\pi(z) = z^2$ is not a covering space.

2. Let $\pi\colon \widetilde{X} \to X$ be a covering space such that $\pi^{-1}(p)$ is finite and nonempty for all $p \in X$. Prove that X is compact Hausdorff if and only if $\widetilde{X}$ is compact Hausdorff.

3. Let X be a Hausdorff space and G be a group acting on X. Assume the following two conditions hold.

   - The action is *free*, i.e. the stabilizer of every point in X is trivial.
   - The action is *properly discontinuous*, i.e. for all $x \in X$, there exists a neighborhood U of x such that the set $\{g \in G \mid g(U) \cap U \neq \varnothing\}$ is finite.

   Prove that the action of G on X is a covering space action.

   Remark 0.1. The second condition is immediate if G is finite, so this implies that all free actions of finite groups on Hausdorff spaces are covering space actions.

4. Let $f\colon \widetilde{X} \to X$ be a covering space.

   (a) If $g\colon Y \to X$ is a continuous map, then define
   $$g^*(\widetilde{X}) = \{(y,p) \mid g(y) = f(p)\} \subset Y \times \widetilde{X}.$$
   Also, let $g^*(f)\colon g^*(\widetilde{X}) \to Y$ be the restriction of of the projection $Y \times \widetilde{X} \to Y$ onto the first factor. Prove that $g^*(f)\colon g^*(\widetilde{X}) \to Y$ is a covering space.

   (b) If $g\colon X \to X$ is the identity map, then prove that $g^*(\widetilde{X})$ is isomorphic to $\widetilde{X}$.

   (c) If X' is a subspace of X and $g\colon X' \to X$ is the inclusion of X' into X, prove that $g^*(f)\colon g^*(\widetilde{X}) \to X'$ is isomorphic to the restriction of f to X'.

   (d) If $\widetilde{X}$ is a trivial cover of X and $g\colon Y \to X$ is a continuous map, prove that $g^*(\widetilde{X})$ is a trivial cover of Y.

   (e) If $g\colon Y \to X$ and $h\colon Z \to Y$ are continuous maps, prove that the cover $(g \circ h)^*(\widetilde{X})$ of Z is isomorphic to $h^*(g^*(\widetilde{X}))$ of Z.

   (f) Let $g\colon Y \to X$ is the constant map that takes every point of Y to a fixed point $p_0 \in X$, prove that $g^*(\widetilde{X})$ is a trivial cover of Y. Hint: You can prove this directly, but it is better to deduce it from the last two parts of the exercise.