Math 60330: Basic Geometry and Topology
Problem Set 4

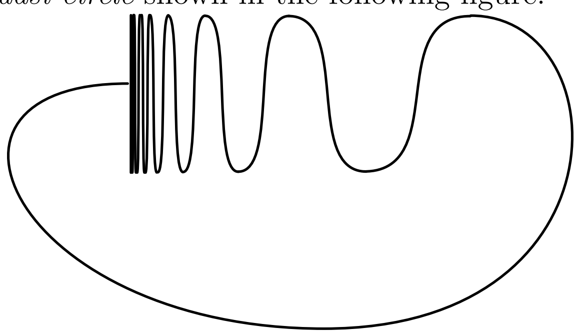
1. Construct the universal cover of the following space:

$$X = \{x \in \mathbb{R}^3 \mid \|x\| = 1\} \cup \{(x, 0, 0) \in \mathbb{R}^3 \mid -1 \leqslant x \leqslant 1\}.$$

   Hint: to prove that the resulting space is simply-connected, you will need to use the next-to-last exercise from the previous problem set.

2. Let X be the subspace of $\mathbb{R}^2$ consisting of the four sides of the square $[0, 1]\times[0, 1]$ together with the segments of the vertical lines $x = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ inside the square. Show that for every covering space $\rho : \tilde{X} \to X$, there is some neighborhood of the left edge of X that lifts homeomorphically to $\tilde{X}$. Deduce that X has no simply-connected covering space.

3. Let Y be the *quasi-circle* shown in the following figure:



   Thus Y is a closed subspace of $\mathbb{R}^2$ consisting of the union of the set

$$\{(x, y) \in \mathbb{R}^2 \mid 0 < x \leqslant 1,\, y = \sin(\frac{1}{x})\},$$

   the set

$$\{(0, y) \in \mathbb{R}^2 \mid -1 \leqslant y \leqslant 1\},$$

   and an arc joining the point $(0, 0)$ to the point $(1, \sin(1))$. Collapsing the segment of Y in the y-axis to a point gives a quotient map $f : Y \to S^1$. Show that f does not lift to the universal covering space $\rho : \mathbb{R} \to S^1$ even though $\pi_1(Y) = 0$. Thus local path-connectedness of Y is a necessary hypothesis in the lifting criterion.

4. Let X be a path-connected, locally path-connected space with $\pi_1(X)$ a finite group. Prove that every map $f : X \to S^1$ is nullhomotopic.

5. Let $f : Y \to X$ be a simply-connected covering space of X, let $A \subset X$ be a path-connected, locally path-connected subspace, and let $B \subset Y$ be a path component of $f^{-1}(A)$. Prove that $f|_B : B \to A$ is the covering space corresponding to the kernel of the map $\pi_1(A) \to \pi_1(X)$.