Math 60330: Basic Geometry and Topology
Problem Set 5

1. Let Σ_g be an oriented genus g surface with a basepoint $p \in \Sigma_g$. Assume that $g \geqslant 2$. Prove that $\pi_1(\Sigma_g, p)$ is not abelian. Hint : find a surjective homomorphism from $\pi_1(\Sigma_g, p)$ to the dihedral group of order 8.

2. Let X be a connected graph with vertex set $V(X)$ and edge set $E(X)$. Assume that both $V(X)$ and $E(X)$ are finite sets.

   (a) If X is a tree, prove that $|V(X)| - |E(X)| = 1$.

   (b) For $p \in V(X)$, prove that $\pi_1(X, p)$ is a free group of rank r with $|V(X)| - |E(X)| = 1 - r$.

   (c) Let F_n be a free group of rank n and let $G \subset F_n$ be a subgroup. As we showed in class, G is a free group; let m be its rank. Assume that $r = [F_n : G]$ is finite. Find a formula for m in terms of n and r.

3. Let $T^2 = S^1 \times S^1$ and let X be the quotient of $T^2 \sqcup T^2$ that identifies the circles $S^1 \times 1$ in both tori homeomorphically. Calculate the fundamental group of X.

4. Let $W = S^1 \vee S^1$ and let $p \in W$ be the wedge point. Identify $\pi_1(W, p)$ with the free group on a and b, where a goes around one S^1 and b goes around the other one. Construct three connected 4-fold covers of W that are distinct up to covering space equivalence, including at least 1 irregular cover. For each of these three covers, describe the covering map, say whether or not the cover is regular, and give a free basis (in terms of a and b) for the corresponding subgroup of $\pi_1(W, p)$.

5. Let $\{p_1, \ldots, p_n\}$ be a set of n distinct points in S^2 and let X be the quotient space of S^2 that identifies all the p_i to a single point. Let $q \in X$ be a basepoint. Calculate $\pi_1(X, q)$.