Math 60330: Basic Geometry and Topology
Problem Set 6

1. Consider the following two atlases for S^n:

   - The atlas
   $$\mathcal{A} = \{\phi_{x_i>0}\colon U_i \to V,\, \phi_{x_i<0}\colon U_i \to V \mid 1 \leqslant i \leqslant n+1\},$$
   where
   $$\begin{aligned} U_{x_i>0} &= \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_i > 0\}, \\ U_{x_i<0} &= \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_i < 0\}, \\ V &= \{(y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_1^2 + \cdots + y_n^2 < 1\} \end{aligned}$$
   and
   $$\begin{aligned} \phi_{x_i>0}(x_1, \ldots, x_{n+1}) &= (x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1}), \\ \phi_{x_i<0}(x_1, \ldots, x_{n+1}) &= (x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1}). \end{aligned}$$

   - The atlas
   $$\mathcal{A}' = \{\phi_\infty\colon S^n \backslash \{(0,0,1)\} \to \mathbb{R}, \phi_{-\infty}\colon S^n \backslash \{(0,0,-1)\} \to \mathbb{R}$$
   where
     - ϕ_∞ take $p \in S^n \backslash \{(0,0,1)\}$ to the intersection of the line from p to $(0,0,1)$ with $\mathbb{R}^n \subset \mathbb{R}^{n+1}$, and
     - $\phi_{-\infty}$ takes $p \in S^n \backslash \{(0,0,-1)\}$ to the intersection of the line from p to $(0,0,-1)$ with $\mathbb{R}^n \subset \mathbb{R}^{n+1}$.

   We proved in class that $\mathcal{A}$ was a smooth atlas

   (a) Prove that $\mathcal{A}'$ is also a smooth atlas.

   (b) Prove directly that $\mathcal{A}$ and $\mathcal{A}'$ are equivalent smooth atlases.

2. Define two different smooth atlases on $\mathbb{R}$:

   - The atlas $\mathcal{A}$ has a single chart $\phi : U \to V$ with $U = V = \mathbb{R}$ and $\phi(x) = x$.
   - The atlas $\mathcal{A}$ has a single chart $\phi' : U' \to V'$ with $U' = V' = \mathbb{R}$ and $\phi'(x) = x^3$.

   This gives two different smooth manifolds $(\mathbb{R}, \mathcal{A})$ and $(\mathbb{R}, \mathcal{A}')$ whose underlying set is $\mathbb{R}$. Prove the following things:

   (a) The identity map $i : \mathbb{R} \to \mathbb{R}$ is a smooth homeomorphism from $(\mathbb{R}, \mathcal{A})$ to $(\mathbb{R}, \mathcal{A}')$, but is not a diffeomorphism (here recall that a diffeomorphism is a smooth homeomorphism whose inverse is also smooth).

(b) Prove that exists some $j : \mathbb{R} \to \mathbb{R}$ which is a smooth diffeomorphism from $(\mathbb{R}, \mathcal{A})$ to $(\mathbb{R}, \mathcal{A}')$.

Remark 0.1. In fact, one can prove that any two smooth atlases on $\mathbb{R}$ yields diffeomorphic smooth manifolds. Even more is true: for any manifold of dimension at most 3, any two smooth atlases yield diffeomorphic smooth manifolds (one says that there are no "exotic" smooth structures in these dimensions). Remarkably, this starts to fail in dimension 4; in fact, there exist uncountably many non-diffeomorphic smooth structures on $\mathbb{R}^4$.

3. Let $f\colon M_1^{n_1} \to M_2^{n_2}$ be a smooth map between smooth manifolds. Consider $p \in M_1^{n_1}$. Recall that in class we defined the derivative map $D_p f\colon T_p M_1^{n_1} \to T_{f(p)} M_2^{n_2}$ as follows. Let $\phi\colon U \to V$ be a chart for $M_1^{n_1}$ with $p \in U$ and let $\phi'\colon U' \to V'$ be a chart for $M_2^{n_2}$ with $f(U) \subset U'$. Then $D_p f\colon T_p M_1^{n_1} \to T_{f(p)} M_2^{n_2}$ is the composition

$$T_p M_1^{n_1} = T_{\phi(p)} V \to T_{\phi'(f(p))} V' = T_{f(p)} M_2^{n_2},$$

where the map

$$T_{\phi(p)} V \to T_{\phi'(f(p))} V'$$

is the linear map obtained by taking the derivative at $\phi(p)$ of the smooth map $V \to V'$ that equals

$$V \xrightarrow{\phi^{-1}} U \xrightarrow{f} U' \xrightarrow{\phi'} V'.$$

This makes sense since V and V' are open subsets of Euclidean space, so we can take this derivative as in ordinary calculus.

- **Problem**: Prove that this does not depend on our choice of charts (this is mostly a matter of coming to grips with the various identifications we have made, though at some point you will use the chain rule).