Math 60330: Basic Geometry and Topology
Problem Set 7

1. Let M^n be a smooth manifold and let $p \in M^n$. Let $C^\infty(M^n)$ be the ring of all smooth functions $M^n \to \mathbb{R}$.

   (a) For $\vec{v} \in T_p M^n$, define $\nabla_{\vec{v}} \colon C^\infty(M^n) \to \mathbb{R}$ by letting $\nabla_{\vec{v}}(f)$ equal the image of $\vec{v}$ under the map
   $$T_p M^n \xrightarrow{D_p f} T_{f(p)}\mathbb{R} = \mathbb{R}.$$
   Prove that this satisfies the following properties.

      i. For $\vec{v} \in T_p M^n$ and $f, g \in C^\infty(M^n)$, we have
      $$\nabla_{\vec{v}}(f+g) = \nabla_{\vec{v}}(f) + \nabla_{\vec{v}}(g).$$

      ii. For $\vec{v} \in T_p M^n$ and $f, g \in C^\infty(M^n)$, we have the Leibniz rule
      $$\nabla_{\vec{v}}(fg) = g(p)\nabla_{\vec{v}}(f) + f(p)\nabla_{\vec{v}}(g).$$

   (b) Now assume that $\Psi \colon C^\infty(M) \to \mathbb{R}$ is a map such that
   $$\Psi(f+g) = \Psi(f) + \Psi(g).$$
   and
   $$\Psi(fg) = g(p)\Psi(f) + f(p)\Psi(g)$$
   for all $f, g \in C^\infty(M^n)$. Prove that there exists a unique $\vec{v} \in T_p M^n$ such that $\Psi(f) = \nabla_{\vec{v}}(f)$ for all $f \in C^\infty(M^n)$.

   Remark 0.1. Maps $\Psi \colon C^\infty(M) \to \mathbb{R}$ satisfying the above two properties are called *derivations at p*. The above exercise shows that you can identify the tangent space of M^n at p as the set of derivations at p.

2. (a) Let $M^n \subset \mathbb{R}^m$ be a smooth submanifold and let $p \in M^n$. Let $\phi \colon U \to V$ be a diffeomorphism from an open set $U \subset M^n$ to an open set $V \subset \mathbb{R}^n$ such that $p \in U$. Prove that the image of
   $$D_{\phi(p)}(\phi^{-1}) \colon T_{\phi(p)}V = \mathbb{R}^n \longrightarrow T_p\mathbb{R}^m = \mathbb{R}^m$$
   is independent of the choice of chart. As we discussed in class this allows us to regard TM^n as a subset of $\mathbb{R}^m \times \mathbb{R}^m$.

   (b) If $S^n \subset \mathbb{R}^{n+1}$ is the standard embedding of S^n into $\mathbb{R}^{n+1}$, then prove that under the above identification TS^n equals the subset
   $$\{(p, \vec{v}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|p\| = 1 \text{ and } p \cdot \vec{v} = 0\},$$
   where $\cdot$ is the dot product.

(c) Let $M_1^{n_1} \subset \mathbb{R}^{m_1}$ and $M_2^{n_2} \subset \mathbb{R}^{m_2}$ be two smooth submanifolds and let $f\colon M_1^{n_1} \to M_2^{n_2}$ be a map that is smooth in the following sense: for $p \in M_1^{n_1}$, there exists an open neighborhood $W \subset \mathbb{R}^{m_1}$ of p and a smooth function $F\colon W \to \mathbb{R}^{m_2}$ such that $F|_{W \cap M_1^{n_1}} = f|_{W \cap M_1^{n_1}}$ (this is equivalent to the definition of a smooth map between $M_1^{n_1}$ and $M_2^{n_2}$, though we don't have the tools to prove this yet). Prove that $D_p f\colon T_p M_1^{n_1} \to T_{f(p)} M_2^{n_2}$ equals the restriction of $D_p F\colon T_p \mathbb{R}^{m_1} \to T_{f(p)} \mathbb{R}^{m_2}$ to $T_p M_1^{n_1}$.

3. Fix some real numbers $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{n+1}$. Regarding S^n as a subspace of $\mathbb{R}^{n+1}$, define a map $f : S^n \to \mathbb{R}$ via the formula

$$f(x_1, \ldots, x_{n+1}) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_{n+1} x_{n+1}^2 \qquad \text{for } (x_1, \ldots, x_{n+1}) \in S^n \subset \mathbb{R}^{n+1}.$$

Say that a point $p \in S^n$ is a *regular point* of f if the derivative map $D_p f\colon T_p S^n \to T_{f(p)} \mathbb{R}$ is surjective. The *regular values* of f are the set of all $x \in \mathbb{R}$ such that all points of $f^{-1}(x)$ are regular points of f. **Problem**: Prove that the regular values of f are exactly the set $\mathbb{R} \backslash \{\lambda_1, \ldots, \lambda_{n+1}\}$.

4. Let G be a Lie group, that is, a group G that is also a smooth manifold such that the multiplication map $M \times M \to M$ taking (x, y) to xy and the inversion map $M \to M$ taking x to x^{-1} are smooth. Letting n be the dimension of G, prove that $TG \cong G \times \mathbb{R}^n$.