Math 60330: Basic Geometry and Topology
Problem Set 9

1. Define a 1-form ω on $M^2 = \mathbb{R}^2 \backslash \{0\}$ via the formula

$$\omega = \left(\frac{-y}{x^2+y^2}\right) \mathrm{dx} + \left(\frac{x}{x^2+y^2}\right) \mathrm{dy}.$$

   (a) Let $\gamma\colon [0,1] \to M^2$ be a circle of radius $r > 0$ around $(0,0)$. Calculate $\int_\gamma \omega$.

   (b) Prove that there does not exist some smooth function $f\colon M^2 \to \mathbb{R}$ such that $\omega = \mathrm{df}$.

   (c) Define $M_2^2 = \{(x,y) \mid x > 0\}$. Construct an explicit function $f\colon M_2^2 \to \mathbb{R}$ such that $\omega = \mathrm{df}$.

2. (a) Let M^n be a smooth manifold and let $\omega \in \Omega^1(M^n)$ be such that $\omega = \mathrm{df}$ for some smooth function $f\colon M^n \to \mathbb{R}$. If $\gamma\colon [a,b] \to M^n$ is a closed path (i.e. a path such that $\gamma(a) = \gamma(b)$), prove that $\int_\gamma \omega = 0$.

   (b) Let M^n be a smooth connected manifold and let $\omega \in \Omega^1(M^n)$. Assume that for all closed paths $\gamma\colon [a,b] \to M^n$, we have $\int_\gamma \omega = 0$. The goal of this problem is to prove the converse of part a, i.e. that there exists some smooth function $f\colon M^n \to \mathbb{R}$ such that $\omega = \mathrm{df}$.

      i. Let $\gamma_1\colon [0,1] \to M^n$ and $\gamma_2\colon [0,1] \to M^n$ be two smooth paths such that $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$. Prove that $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$. Warning: you have to be careful; it is not necessarily the case that the path $\delta\colon [0,2] \to M^n$ defined via the formula

$$\delta(t) = \begin{cases} \gamma_1(t) & \text{if } 0 \leqslant t \leqslant 1, \\ \gamma_2(t-1) & \text{if } 1 \leqslant t \leqslant 2 \end{cases}$$

      is smooth. The problem occurs at $t = 1$. Try to reparameterize your paths such that this is smooth.

      ii. Fix some basepoint $x_0 \in M^n$. Define a function $f\colon M^n \to \mathbb{R}$ by letting $f(p) = \int_\gamma \omega$, where $\gamma\colon [0,1] \to M^n$ is a path such that $\gamma(0) = x_0$ and $\gamma(1) = p$ (this is well-defined by part a). Prove that $\mathrm{df} = \omega$.

3. Let $\omega \in \mathcal{A}^n(\mathbb{R}^n)$ and let M be an $n \times n$ matrix. Prove that for all $\vec{v}_1, \ldots, \vec{v} \in \mathbb{R}^n$, we have

$$\omega(M(\vec{v}_1), \ldots, M(\vec{v}_n)) = \det(M)\omega(\vec{v}_1, \ldots, \vec{v}_n).$$

4. Let V be a vector space and let $\omega_1, \ldots, \omega_k \in V^* = \mathcal{A}^1(V)$. Prove that the ω_i are linearly independent if and only if $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k = 0$.

5. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets and let $f \colon U \to V$ be a smooth map. Consider $\omega \in \Omega^k(V)$. Let $x_1, \ldots, x_n$ be the coordinates on $\mathbb{R}^n$ and let $y_1, \ldots, y_m$ be the coordinates on $\mathbb{R}^m$. Set

$$\mathcal{I} = \{(i_1, \ldots, i_k) \mid 1 \leqslant i_1 < \cdots < i_k \leqslant n\}$$

and

$$\mathcal{J} = \{(j_1, \ldots, j_k) \mid 1 \leqslant j_1 < \cdots < j_k \leqslant m\}$$

Write

$$\omega = \sum_{J \in \mathcal{J}} g_J \mathrm{dy}_J \quad \text{and} \quad f^*(\omega) = \sum_{I \in \mathcal{I}} h_I \mathrm{dx}_I.$$

State and prove a relationship between f, the g_J, and the h_I.