Math 60440: Basic Topology II
Problem Set 4

1. Let A be a subspace of a space X. Assume that A is a retract of X. Prove that the map $\mathrm{H}_n(A) \to \mathrm{H}_n(X)$ induced by the inclusion $A \hookrightarrow X$ is an injection.

2. Prove that being chain homotopic is an equivalence relation on maps between chain complexes.

3. Prove that the relative homology group $\mathrm{H}_1(\mathbb{R}, \mathbb{Q})$ is free abelian, and identify a basis for it.

4. (a) Calculate all the homology groups of an orientable surface of genus g by first constructing a CW complex structure and then using the cellular chain complex (we don't know an explicit description of the boundary maps for this, so you'll have to figure them out from what we know about H_1 and H_0).

   (b) Do the same for the nonorientable surface of genus g (i.e. the connect sum of g copies of $\mathbb{RP}^1$).

5. (a) Using the cellular chain complex, prove that if X is an n-dimensional CW complex, then $\mathrm{H}_n(X)$ is free abelian.

   (b) Let A be a finitely generated free abelian group and let B be a finitely generated abelian group. Construct a two-dimensional CW complex X such that

$$\mathrm{H}_k(X) = \begin{cases} A & \text{if } k = 2, \\ B & \text{if } k = 1, \\ \mathbb{Z} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

6. Consider a commutative diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"'] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"'] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\
B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & B_4 \arrow[r, "\beta_4"] & B_5
\end{tikzcd}$$

   of abelian groups. Assume that both rows are exact. Prove the following:

   (a) If f_2 and f_4 are injective and f_1 is surjective, then f_3 is injective.

   (b) If f_2 and f_4 are surjective and f_5 is injective, then f_3 is surjective.

   (c) If f_2 and f_4 are isomorphisms, f_1 is surjective, and f_5 is injective, then f_3 is an isomorphism.

7. (a) Let

$$0 \longrightarrow A' \xrightarrow{j} A \xrightarrow{q} \overline{A} \longrightarrow 0$$

be a short exact sequence of abelian groups. Prove that the following three statements are equivalent (in these cases, we say that the short exact sequence *splits*):

(i) The map q admits a section, i.e. there exists a homomorphism $s\colon \overline{A} \to A$ such that $q \circ s = \mathrm{id}$.

(ii) The map j admits a retraction, i.e. there exists a homomorphism $r\colon A \to A'$ such that $r \circ j = \mathrm{id}$.

(iii) There exists a commutative diagram of the form

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "j"] \arrow[d, "\mathrm{id}"'] & A \arrow[r, "q"] \arrow[d, "\mathrm{f}"'] & \overline{A} \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "i"] & A' \oplus \overline{A} \arrow[r, "p"] & \overline{A} \arrow[r] & 0
\end{tikzcd}$$

where f is an isomorphism, i is the natural inclusion, and p is the natural surjection.

(b) Give an example of a short exact sequence of abelian groups that does not split.

(c) Let Y be a subspace of a topological space X. Assume that Y is a retract of X. Prove that $\mathrm{H}_n(X) \cong \mathrm{H}_n(Y) \oplus \mathrm{H}_n(X, Y)$ for all n.